Lemma 8. *The Fourier coefficients of F has the following properties:*
(1) If k is even, $F \in A_k(\Gamma; 2r)$ and $\min\{n,m\} < r$, then $a(n,l,m) = 0$.
(2) If k is odd, $F \in A_k(\Gamma; 2r+1)$ and $\min\{n,m\} < r+2$, then $a(n,l,m) = 0$.

Proof. First, we show (1). Assume k is even and $F \in A_k(\Gamma; 2r)$. Put

$$b(n,l,m) := \begin{cases} 2a(n,l,m) & (\text{if } l \neq 0) \\ a(n,0,m) & (\text{if } l = 0) \end{cases}$$

Then for any $n, m \in \mathbb{Z}$ and $t \in \{0, 1, \ldots, r-1\}$, we have

$$\sum_{l=0}^{2\sqrt{nm}} l^{2t} b(n,l,m) = 0$$

It is sufficient to show $b(n,l,m) = 0$ if $\min\{n,m\} < r$. We will show this by induction on $\min\{n,m\}$. If $\min\{n,m\} = 0$, this lemma is trivial. Now we assume that $b(n,l,m) = 0$ if $\min\{n,m\} \leq u < r-1$ and consider the case $\min\{n,m\} = u+1$. From Lemma 7, $b(n,l,m) = 0$ if $l > u+1$. Then we have

$$\sum_{l=0}^{u+1} l^{2t} b(n,l,m) = 0$$

for any $t \in \{0, 1, \ldots, r-1\}$. Hence, by the Vandermonde formula, we have $b(n,l,m) = 0$.

Second, we show (2). Assume k is odd and $F \in A_k(\Gamma; 2r+1)$. Put $b(n,l,m) := 2a(n,l,m)$. We remark that $a(m,l,m) = 0$, $a(n,n,m) = 0$ and $a(n,m,m) = 0$. Then for any $n, m \in \mathbb{Z}$ and $t \in \{0, 1, \ldots, r-1\}$, we have

$$\sum_{l=1}^{2\sqrt{nm}} l^{2t+1} b(n,l,m) = 0$$

It is sufficient to show $b(n,l,m) = 0$ if $\min\{n,m\} < r$. We will show this by induction on $\min\{n,m\}$. If $\min\{n,m\} = 0$, this lemma is trivial. Now we assume that $b(n,l,m) = 0$ if $\min\{n,m\} \leq u < r+1$ and consider the case $\min\{n,m\} = u+1$. From Lemma 6, $b(n,l,m) = 0$ if $l > u+1$. Then we have

$$\sum_{l=1}^{u+1} l^{2t+1} b(n,l,m) = 0$$

for any $t \in \{0, 1, \ldots, r-1\}$. Hence, by the Vandermonde formula, we have $b(n,l,m) = 0$. □