Corollary 5. *There exist two exact sequences.*
(1) If k is even, $A_k(\Gamma) = A_k(\Gamma; 0)$ and

$$0 \longrightarrow A_k(\Gamma; 2r+2) \longrightarrow A_k(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma}).$$

(2) If k is odd, $A_k(\Gamma) = A_k(\Gamma; 1)$ and

$$0 \longrightarrow A_k(\Gamma; 2r+3) \longrightarrow A_k(\Gamma; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{M}^{\mathrm{skew}}_{k+2r+1}(\widetilde{\Gamma}).$$

To study the image $D_r(A_k(\Gamma; r))$ more precisely, we will investigate the Fourier coefficients of modular forms. Let $F \in A_k(\Gamma)$. Put the Fourier coefficients of F by

$$F(Z) = \sum_{n,l,m \in \mathbb{Z}} a(n,l,m) q^n \zeta^l p^m.$$

Because

$$(D_r(F))(\tau, \omega) := \sum_{n,m \in \mathbb{Z}} \left(\sum_{l \in \mathbb{Z}} \left(2\pi\sqrt{-1}l\right)^r a(n,l,m) \right) q^n p^m,$$

if $F \in A_k(\Gamma; r)$, for any $n \in \mathbb{Z}, m \in \mathbb{Z}$ and $t < r$,

$$\sum_{l \in \mathbb{Z}} l^t a(n,l,m) = 0.$$

Lemma 6. *The Fourier coefficients of F satisfy the following equations:*
(1) $a(n,-l,m) = (-1)^k a(n,l,m)$.
(2) $a(m,l,n) = (-1)^k a(n,l,m)$.
(3) $a(n + xl + x^2 m, l + 2xm, m) = a(n,l,m)$ for any $x \in \mathbb{Z}$.
(4) $a(n, l + 2xn, m + xl + x^2 n) = a(n,l,m)$ for any $x \in \mathbb{Z}$.
(5) If k is odd, then $a(n,0,m) = 0$ and $a(n,l,n) = 0$.
(6) If $4nm - l^2 < 0$, $n < 0$ or $m < 0$, then $a(n,l,m) = 0$.

Proof. The equations (1)-(5) are easy to show from the transformation formula of modular forms. The equation (6) is well-known as Köcher principle. □

Next lemma is easy, but this is a key of our story.

Lemma 7. *If $|l| > \min\{n,m\}$ and $a(n,l,m) \neq 0$, there exist n', l', m' such that $\min\{n', m'\} < \min\{n,m\}$ and $a(n',l',m') \neq 0$.*

Proof. It is obvious from Lemma 6 (3)(4). □