operator $D_r : \mathrm{Hol}(\mathbb{H}_2, \mathbb{C}) \to \mathrm{Hol}(\mathbb{H}^2, \mathbb{C})$ by

$$(D_r(F))(\tau,\omega) := \left(\frac{\partial^r F}{\partial z^r}\right)\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}.$$

and put

$$A_k(\Gamma; r) := \{\, F \in A_k(\Gamma) \mid D_t(F) = 0 \text{ for any } t < r\}.$$

We remark that there is a descent sequence of vector spaces

$$A_k(\Gamma) = A_k(\Gamma;0) \supset A_k(\Gamma;1) \supset A_k(\Gamma;2) \supset A_k(\Gamma;3) \supset \cdots$$

and

$$\bigcap_{r\in\mathbb{N}_0} A_k(\Gamma;r) = \{0\}.$$

Lemma 3. *There exists an exact sequence*

$$0 \longrightarrow A_k(\Gamma; r+1) \longrightarrow A_k(\Gamma;r) \xrightarrow{D_r} \mathrm{Hol}(\mathbb{H}^2, \mathbb{C}).$$

This lemma insists that, if we can know the dimension of $D_r(A_k(\Gamma;r))$ possibly, we have the dimension of $A_k(\Gamma)$ by

$$\dim_{\mathbb{C}} A_k(\Gamma) = \sum_{r=0}^{\infty} \dim_{\mathbb{C}} D_r(A_k(\Gamma;r)).$$

Indeed, from the next section, we will calculate the upper bound of the dimension of $D_r(A_k(\Gamma;r))$. Hence we will have the upper bound of the dimension of $A_k(\Gamma)$. Therefore, by constructing sufficiently many modular forms, we can show this upper bound is the true dimension of $A_k(\Gamma)$.

3.4 Estimation

The following lemma is easy to show from the transformation formula of modular forms.

Lemma 4. *The image by D_r has the following properties.*
(1) If k is even and if r is even, $D_r(A_k(\Gamma;r)) \subset \mathrm{M}^{\mathrm{sym}}_{k+r}(\widetilde{\Gamma})$.
(2) If k is even and if r is odd, $D_r(A_k(\Gamma;r)) = \{0\}$.
(3) If k is odd and if r is even, $D_r(A_k(\Gamma;r)) = \{0\}$.
(4) If k is odd and if r is odd, $D_r(A_k(\Gamma;r)) \subset \mathrm{M}^{\mathrm{skew}}_{k+r}(\widetilde{\Gamma})$.

Hence we can improve the exact sequence given by Lemma 3.