conditions:
(1) For any fixed $\omega_0 \in \mathbb{H}$, the function $f(\tau, \omega_0)$ on $\tau \in \mathbb{H}$ belongs to $\mathrm{M}_k(\widetilde{\Gamma})$.
(2) For any fixed $\tau_0 \in \mathbb{H}$, the function $f(\tau_0, \omega)$ on $\omega \in \mathbb{H}$ belongs to $\mathrm{M}_l(\widetilde{\Gamma})$.
We denote by $\mathrm{M}_{k,l}(\widetilde{\Gamma})$ the space of all Witt modular forms of weight (k, l) with respect to $\widetilde{\Gamma}$. For $r, s \in \mathbb{N} \cup \{0\}$, define subspaces of $\mathrm{M}_{k,l}(\widetilde{\Gamma})$ by

$$\mathrm{M}_{k,l}(\widetilde{\Gamma}; r, s) := \left\{ f \in \mathrm{M}_{k,l}(\widetilde{\Gamma}) \;\middle|\; \begin{array}{ll} f(\tau, \omega_0) \in \mathrm{M}_k(\widetilde{\Gamma}; r) & \text{for any } \omega_0 \in \mathbb{H} \\ f(\tau_0, \omega) \in \mathrm{M}_l(\widetilde{\Gamma}; s) & \text{for any } \tau_0 \in \mathbb{H} \end{array} \right\}.$$

By Witt [Wi, Satz A], we have

$$\mathrm{M}_{k,l}(\widetilde{\Gamma}; r, s) = \mathrm{M}_k(\widetilde{\Gamma}; r) \otimes_{\mathbb{C}} \mathrm{M}_l(\widetilde{\Gamma}; s).$$

Hence its Poincaré series is given by

$$\begin{aligned} P_{(\widetilde{\Gamma}; r, s)}(x, y) :=& \sum_{k,l \in \mathbb{N} \cup \{0\}} \left(\dim_{\mathbb{C}} \mathrm{M}_{k,l}(\widetilde{\Gamma}; r, s) \right) x^k y^l \\ =& P_{(\widetilde{\Gamma}; r)}(x) P_{(\widetilde{\Gamma}; s)}(y) \\ =& \frac{x^{12r} y^{12s}}{(1 - x^4)(1 - x^6)(1 - y^4)(1 - y^6)}. \end{aligned}$$

Put $\mathrm{M}_{k,l}(\widetilde{\Gamma}; r) := \mathrm{M}_{k,l}(\widetilde{\Gamma}; r, r)$. We say $f \in \mathrm{M}_{k,k}(\widetilde{\Gamma}; r)$ is symmetric or skew-symmetric if $f(\tau, \omega) = f(\omega, \tau)$ or $f(\tau, \omega) = -f(\omega, \tau)$ and denote by $f \in \mathrm{M}^{\mathrm{sym}}_{k,k}(\widetilde{\Gamma}; r)$ or $f \in \mathrm{M}^{\mathrm{skew}}_{k,k}(\widetilde{\Gamma}; r)$, respectively. The structure of these spaces are easily determined. Their Poincaré series are given by

$$\begin{aligned} P^{\mathrm{sym}}_{(\widetilde{\Gamma}; r)}(x) :=& \sum_{k \in \mathbb{N} \cup \{0\}} \left(\dim_{\mathbb{C}} \mathrm{M}^{\mathrm{sym}}_{k,k}(\widetilde{\Gamma}; r) \right) x^k \\ =& \frac{x^{12r}}{(1 - x^4)(1 - x^6)(1 - x^{12})}, \\ P^{\mathrm{skew}}_{(\widetilde{\Gamma}; r)}(x) :=& \sum_{k \in \mathbb{N} \cup \{0\}} \left(\dim_{\mathbb{C}} \mathrm{M}^{\mathrm{skew}}_{k,k}(\widetilde{\Gamma}; r) \right) x^k \\ =& \frac{x^{12(r+1)}}{(1 - x^4)(1 - x^6)(1 - x^{12})}. \end{aligned}$$

3.3 Differential operator

For a complex domain X, we denote by $\mathrm{Hol}(X, \mathbb{C})$ the set of all holomorphic functions from X to $\mathbb{C}$. For $r \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, define a differential