3 Proof

For the sake of simplicity, in this exposition, we give a proof only on the simplest case: scalar valued full modular case. Hence, from now on, we assume $\Gamma := \mathrm{Sp}(2,\mathbb{Z})$ and $s = 0$. But we insist that our proof is available for all cases in **Theorem 1** and **Theorem 2**.

Anyway, to prove the theorem, we prepare some notations. Let $\widetilde{\Gamma} := \mathrm{SL}(2,\mathbb{Z})$, $q := \mathrm{e}(\tau) := \exp(2\pi\sqrt{-1}\tau)$, $\zeta := \mathrm{e}(z)$ and $p := \mathrm{e}(\omega)$.

3.1 Elliptic modular forms

We denote the complex upper half plane by

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}.$$

For a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ and $k \in \mathbb{Z}$, we say f is an elliptic modular form of weight k with respect to $\widetilde{\Gamma}$ if f satisfies the following two conditions:

(1) For any $M \in \widetilde{\Gamma}$, $f|_k M = f$.

(2) f is bounded at all the cusps.

Let $a(n)$ be the Fourier coefficients of f defined by

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n.$$

We denote by $\mathrm{M}_k(\widetilde{\Gamma})$ the space of all elliptic modular forms of weight k with respect to $\widetilde{\Gamma}$. Put $\mathrm{M}_*(\widetilde{\Gamma}) := \bigoplus_{k\in\mathbb{Z}} \mathrm{M}_k(\widetilde{\Gamma})$. The space $\mathrm{M}_*(\widetilde{\Gamma})$ is a graded ring. For $r \in \mathbb{N} \cup \{0\}$, define subspaces of $\mathrm{M}_k(\widetilde{\Gamma})$ by

$$\mathrm{M}_k(\widetilde{\Gamma}; r) := \left\{ f \in \mathrm{M}_k(\widetilde{\Gamma}) \;\middle|\; a(n) = 0 \text{ if } n < r \right\}.$$

the structure of $\mathrm{M}_*(\widetilde{\Gamma})$ is already known. Namely, the graded ring $\mathrm{M}_*(\widetilde{\Gamma})$ is generated by algebraically independent two modular forms of weight 4 and 6. Its Poincaré series is given by

$$P_r(x) := \sum_{k\in\mathbb{N}\cup\{0\}} \left(\dim_{\mathbb{C}} \mathrm{M}_k(\widetilde{\Gamma}; r)\right) x^k := \frac{x^{12r}}{(1-x^4)(1-x^6)}.$$

3.2 Witt modular forms

For a holomorphic function $f : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ and $k, l \in \mathbb{Z}$, we say f is a Witt modular form of weight (k, l) with respect to $\widetilde{\Gamma}$ if f satisfies the following two