where we denote by ψ_3 the character defined by $\psi_3(M) = \left(\frac{-3}{\det(D)}\right)$ and by ψ_4 the character defined by $\psi_4(M) = \left(\frac{-1}{\det(D)}\right)$.

In these cases, the structure of $A_*(\Gamma)$ is already known.

Theorem 1. *For each* $\Gamma = \mathrm{Sp}(2,\mathbb{Z}), \Gamma_0(2), \Gamma_{0,\psi_3}(3)$ *or* $\Gamma_{0,\psi_4}(4)$*, the graded ring* $A_*(\Gamma)$ *is generated by five modular forms. First four generators are algebraically independent and the square of the last generator is in the subring generated by first four.*

Γ	The weights of first four generators	The weights of the last generators	References
$\mathrm{Sp}(2,\mathbb{Z})$	$4, 6, 10, 12$	35	Igusa [Ig1, Ig2]
$\Gamma_0(2)$	$2, 4, 4, 6$	19	Ibukiyama [Ib1]
$\Gamma_{0,\psi_3}(3)$	$1, 3, 3, 4$	14	Ibukiyama [Ib1] Aoki-Ibukiyama [AI]
$\Gamma_{0,\psi_4}(4)$	$1, 2, 2, 3$	11	Hayashida-Ibukiyama [HI]

We remark that, in all cases, the last generators are obtained from the first four using by Rankin-Cohen-Ibukiyama differential operators in [AI].

2.2 Vector valued case

Let s be a non-negative integer, V be a $(s+1)$-dimensional $\mathbb{C}$-vector space and $\rho : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{GL}(V)$ be a rational representation. It is well-known that ρ is a rational irreducible representation if and only if $\rho = \rho_{k,s} := \mathrm{Sym}^s \otimes \det^k$. For the sake of simplicity, in this exposition, we fix a coordinate of $\mathrm{Sym}^s \otimes \det^k$. Namely, put $V := \mathbb{C}^{s+1}$ and $\rho_{k,s}(A) := (\det A)^k \rho_{0,s}(A)$, where $\rho_{0,s}(A)$ is defined by

$$(u^s, u^{s-1}v, \ldots, v^s) = (x^s, x^{s-1}y, \ldots, y^s)\rho_{0,s}(A) \quad ((u,v) = (x,y)A).$$

For $M \in \mathrm{Sp}(2,\mathbb{R})$ and a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}^{s+1}$, we write

$$(F|_\rho M)(Z) := \rho(CZ+D)^{-1}F(M\langle Z\rangle).$$

Definition 2. *We say F is a Siegel modular forms of weight ρ with respect to Γ if F satisfies the condition $F(Z) = (F|_\rho M)(Z)$ for any $M \in \Gamma$.*