2 Main theorem

2.1 Complex scalar valued case

We denote the Siegel upper half plane of degree 2 by

$$\mathbb{H}_2 := \left\{ Z = {}^tZ = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \;\middle|\; \operatorname{Im} Z > 0 \right\}.$$

The symplectic group

$$\mathrm{Sp}(2,\mathbb{R}) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \;\middle|\; {}^tMJM = J := \begin{pmatrix} O_2 & -E_2 \\ E_2 & O_2 \end{pmatrix} \right\}$$

acts on $\mathbb{H}_2$ transitively by

$$\mathbb{H}_2 \ni Z \mapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \in \mathbb{H}_2.$$

For $M \in \mathrm{Sp}(2,\mathbb{R})$, $k \in \mathbb{Z}$ and a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$, we write

$$(F|_kM)(Z) := \det(CZ+D)^{-k}F(M\langle Z\rangle).$$

Let

$$\mathrm{Sp}(2,\mathbb{Z}) := \mathrm{Sp}(2,\mathbb{R}) \cap \mathrm{M}_4(\mathbb{Z})$$

and $\Gamma \subset \mathrm{Sp}(2,\mathbb{R})$ be a commensurable subgroup with $\mathrm{Sp}(2,\mathbb{Z})$, namely, $\Gamma \cap \mathrm{Sp}(2,\mathbb{Z})$ is a finite index subgroup of Γ and also a finite index subgroup of $\mathrm{Sp}(2,\mathbb{Z})$.

Definition 1. *For a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ and $k \in \mathbb{Z}$, we say F is a Siegel modular forms of weight k with respect to Γ if F satisfies the condition $F(Z) = (F|_kM)(Z)$ for any $M \in \Gamma$.*

We remark that this F is bounded at each cusps by Köcher principle. We denote by $A_k(\Gamma)$ the space of all Siegel modular forms of weight k with respect to Γ. The space $A_*(\Gamma) := \bigoplus_{k\in\mathbb{Z}} A_k(\Gamma)$ is a graded ring.

Put

$$\Gamma_0(N) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{Z}) \;\middle|\; C \equiv O_2 \pmod{N} \right\}$$

for any natural number $N \in \mathbb{N} := \{1,2,3,\dots\}$.

In this exposition, our interest is the case $N = 1,2,3,4$. When $N = 3,4$, we take a character because the structure theorem become simple. That is, for $N = 1,2$, we assume $\Gamma := \Gamma_0(N)$ and for $N = 3,4$, we assume

$$\Gamma := \Gamma_{0,\psi_N}(N) := \left\{ M \in \Gamma_0(N) \;\middle|\; \psi_N(M) = 1 \right\},$$