So to get a little bit the hand on these questions one obviously needs good new ~~methods~~ idea's or at least to possibly generate good ideas tables.

Workstations, Pari, Igusa's theorem, Macaulay + following ~~theorem~~ $\overset{\text{easy}}{\rightsquigarrow}$ to produce tables

<u>Lemma</u> (?)

let $A = q^{-6} \sum_{\substack{r,s \in \mathbb{Z} \\ r \equiv s \\ 2}} s^2 (-1)^r q^{\frac{s^2+r^2}{4}} \cdot y^r$,

$B =$ — (same without s^2)

then

$$(f,g) \mapsto \frac{k}{2} fA - \left(q\frac{d}{dq} f\right)B + g\,B$$

defines

$$I : M_k(\Gamma_1) \oplus S_{k+2}(\Gamma_2) \xrightarrow{\ \simeq\ } J_{k,1}.$$

In particular

$$\chi_4 = VI(E_4, 0)$$
$$\chi_6 = VI(E_6, 0)$$
$$\chi_{10} = VI(0, \Delta)$$
$$\chi_{12} = VI(\Delta, 0).$$

we computed the

Hef $\Upsilon 20, \Upsilon 22, \ldots, \Upsilon 32a, \ldots, \Upsilon 32e$ in $S_{20}^?(\Gamma_2), \to S_{32}^?(\Gamma_2)$

The <u>results</u> in this range:

Yes for 1), 2), 4) (in the range of the computation), 3) yet not finished

|| Interesting phenomena for 5)

$\mathcal{X}_k(\Gamma_2)$ splits over $\mathbb{Q}$ for $k = 24, 26$

$\mathcal{X}_k(\Gamma_2)$ non-split ~~over $\mathbb{Q}$ ... for~~ otherwise ?