(8) (between $D_{F,G}(s)$ and
So what is the relation ~~to~~ the $Z_F(s)$?

A partial answer can be found e.g. in work of Gritsenko

Theorem If $F, G \in S_k^*(\Gamma_2)$ Hef, G Maass-special, then

$$D_{F,G}(s) = \text{cst} \cdot Z_F(s).$$

The proof of these theorems is quite feasible, at least much easier than Andrianov's proof of the functional equation. Note

$F \in S_k(\Gamma_2)$ Hef, $\phi_1 \neq 0$ ~~then~~ $Z_F(s) = c\, D_{F, V\phi_1}$

(i.e. analytic properties of $Z_F(s)$ follow easily!)

So we have

$$\boxed{2) \quad F \in S_k(\Gamma_2) \text{ Hef} \overset{?}{\Rightarrow} \phi_1 \neq 0}$$

Another problem which still remains is

3) $D_{F,F}(s)$ for F Hef, but **not** special?
related to Andr. zeta-fct's of Hefs in $M_k(\Gamma_2)$?

~~Be... result of~~ One has

Theorem (Oda – Evdokimov)
Let $F \in M_k(\Gamma_2)$ Hef. Then $Z_F(s)$ has poles iff F = Maass-special.
Thus, set

$S_k^?(\Gamma_2)$ = span of Hef F in $S_k(\Gamma_2)$ with holomorphic $Z_F(s)$.

Then

$$M_k(\Gamma_2) \underset{\text{compat.}}{\overset{\text{Hecke}}{=}} K S_k(\Gamma_1) \oplus V J_{k,1} \oplus S_k^?(\Gamma_2)$$