Any

$F \in M_k(\Gamma_2)$ has FD: $$F = \sum_{\substack{n,r,m \in \mathbb{Z} \\ \cancel{r^2-4nm \leq 0}}} a_F(n,r,m)\, q^n \zeta^r q'^m$$

$$= \sum_{Q = [n,r,m]} a_F(Q)\, q^n \zeta^r q'^m .$$

One knows

$$a_F(Q) = a_F(Q \circ A) \quad \text{for all } A \in GL_2(\mathbb{Z})$$

$$\left(Q = [n,r,m] = nX^2 + rXY + mY^2, \quad Q \circ A\,(X,Y) = Q\left(\left(A\binom{X}{Y}\right)^t\right) \right)$$

and

Koecher-principle: $a_F(Q) = 0$ unless $Q \geq 0$

Thus

$F \in M_k$ is given by $a_F : \left\{ \begin{array}{l} GL_2(\mathbb{Z})\text{-equiv. classes} \\ \text{of integral, semi-pos.} \\ \text{def. quad forms} \end{array} \right\} \to \mathbb{C}$

As in the elliptic case we have two fundamental problems;

1) Arithmetic nature of the a_F ?

2) How to generate the a_F for a basis $\{F\}$ of M_k ?

To 1) – if one is honest – very few is known though there has been much effort.

What is known

Theorem (Andrianov)

There exist a sequence of natural (Hecke-) operators $T(l)$ ($l = 1, 2, \ldots$) on $M_k(\Gamma_2)$ such that

(1) M_k contains finite basis of simultaneous eigenforms for all $T(l)$

(2) $F \in M_k$ Hecke, $T(n) F = \lambda(n) F$, then $\lambda(n) \in \overline{\mathbb{Q}}$ algebr. integers

(3) $Z_F(s) = \zeta(2s-2k+4) \sum_{l=1}^{\infty} \frac{\lambda(l)}{l^s}$ abs. conv. for $\mathrm{Re}\, s \gg 0$

$Z_F(s) = \prod_p \frac{1}{Q_p(p^{-s})}$, $X^4 Q_p(\frac{1}{X}) =$ monic polynomial of degree 4

(4) $L_F(s)$ has merom. contin. to $\mathbb{C}$

(5) $Z_F^*(s) := (2\pi)^{-2s}\, \Gamma(s)\, \Gamma(s-k+2)\, Z_F(s) = Z_F^*(2k-2-s)$