① Bielefeld, 5.2.91

Elliptic modular forms

Open problems in the theory of Siegel modular forms of degree 2

To start will let me remind you of some basic facts of classical elliptic modular forms. Everybody here has seen at least one elliptic modular forms. Namely, let

$$Q(x_1, \dots, x_r) \in \mathbb{Z}[x_1, \dots, x_r] \quad \text{pos. def. quadr. form,}$$

say

$$\det(Q) = \det\left(\frac{\partial^2 Q}{\partial x_i \partial x_j}\right)_{i,j} = 1,$$

then

$$8 \mid r$$

and

$$\Theta_Q = \sum_{n=0}^{\infty} \#\{\vec{x} \in \mathbb{Z}^r \mid Q(\vec{x}) = n\}\, q^n \in M_{r/2}(\Gamma_1).$$

Here

$$q = e^{2\pi i \tau}, \quad \tau \in \mathfrak{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$$

and

$$M_k(\Gamma_1) = \text{ell. mod. forms of weight } k\ (\in 2\mathbb{Z}) \text{ on } \Gamma_1 := SL_2(\mathbb{Z})$$

$$= \left\{ f : \mathfrak{H} \xrightarrow{\text{hol.}} \mathbb{C} \;\middle|\; \begin{array}{l} a)\ f \text{ per. with per. } 1 \\ b)\ f(-\frac{1}{\tau}) = f(\tau)\,\tau^k \\ c)\ f(\tau) = O(1) \text{ for } \operatorname{Im}\tau \to \infty \end{array} \right\}$$

Recall that

$$a), b) \iff f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1.$$

That's why this Γ_1 comes in.

Any

$$f \in M_k(\Gamma) \text{ has FD: } \quad f = \sum_{n=0}^{\infty} a_f(n)\, q^n.$$

Thus an elliptic modular form is given by its FC's

$$f \longleftrightarrow (a_f(n))_{n \geq 0}$$

and so there arise immediately two natural questions

1) Arithmetic nature of the $a_f(n)$?

2) How to ~~generate~~ (compute) the $a_f(n)$ (say for a basis $\{f\}$ of $M_k(\Gamma_1)$)?