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We have now several examples of automorphic forms. Let's summarize the main ingredients to define automorphic forms:

Lie group	space ~~automorphy factor~~	action	automorphy factor $j(A,x)$	discrete subgroup
$SL_2(\mathbb{R})$	$\mathfrak{H}$	$\tau \to \frac{a\tau+b}{c\tau+d}$	$j(\begin{pmatrix} a & b \\ c & d \end{pmatrix},\tau) = (c\tau+d)^{-k}$	e.g. $SL_2(\mathbb{Z})$
$Sp_n(\mathbb{R})$	$\mathfrak{H}^{(n)} = \{F = F^t \in \mathbb{C}^{2n\times 2n} / \operatorname{Im} F > 0\}$	$\omega \to \alpha\langle\omega\rangle = (A\omega+B)(C\omega+D)^{-1}$, $\alpha = \left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	$\lvert C\omega+D\rvert^{-k}$	$Sp_n(\mathbb{Z})$
	$\mathfrak{H}\times\mathbb{C}$			

regularity condition	space of functions satisfying reg. cond. + $f(\alpha x)\, j(\alpha,x)^{-1} = f(x)$
e.g. holomorphic + ...	elliptic modular forms
~~...~~	Siegel modular forms of degree n
	Jacobi forms

Jacobi forms

~~$J(\mathbb{R}) = SL_2(\mathbb{R})$~~ $H(\mathbb{R}) = \mathbb{R}^2 \cdot S^1 = \{(x,s) \mid x\in\mathbb{R}^2, s\in S^1\}$

with $(x,s)\cdot(y,t) := \left(x+y,\ st\, e^{2\pi i \det\binom{x}{y}}\right)$

$J(\mathbb{R}) = SL_2(\mathbb{R}) \ltimes H(\mathbb{R}) = \{(A,x,s) \mid A\in SL_2(\mathbb{R}),\ (x,s)\in H(\mathbb{R})\}$

with $(A,x,s)(A',x',s') = \left(AA',\ xA'+x',\ ss'\, e^{2\pi i \det\binom{xA'}{x'}}\right)$.

action on $\mathfrak{H}\times\mathbb{C}$:

$$(A,x,s)\cdot(\tau,z) := \left(\frac{a\tau+b}{c\tau+d},\ \frac{z+\lambda\tau+\mu}{c\tau+d}\right) \qquad \left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ x = (\lambda,\mu)\right)$$

factor of automorphy:
given $k, m \in\mathbb{Z}$:

$$j((A,x,s),\tau,z) := (c\tau+d)^{-k}\, e^{2\pi i m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)}\, s^m$$