(6)

The second example is:

Let $L := \{\underset{(x_1,\dots,x_8)}{x} \in \frac{1}{\sqrt{2}} \mathbb{Z}^8 \cup \frac{1}{\sqrt{2}}(\mathbb{Z}^8 + (\frac{1}{2},\dots,\frac{1}{2})) \mid \sum_{i=1}^{8} x_i \in 2\mathbb{Z}\} \subseteq \mathbb{R}^8$.

Exercise: L even, integral, $L = L^\vee$. Hence $\vartheta_L \in M_4(Sl_2(\mathbb{Z}))$.
Compare with the above table:

<u>Corollary</u> $\vartheta_L = E_4$, i.e. $r_L(n) = \#\{x \in L \mid x^2 = 2n\} = 240\,\sigma_3(n)$.
You admit that this is by no means trivial.

Interesting sequences may be indexed by other sets aside from the natural numbers, e.g. by symmetric 2×2 matrices:

$$r_L\left(\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}\right) = \{(x,y) \in L^2 \mid x^2 = n,\ y^2 = m,\ x\cdot y = r\} \quad (n,\dots,r \in \mathbb{Z})$$

generating function:

$$\vartheta_L^{(2)}(\omega) = \sum_{T = T^t} a(T)\, e^{2\pi i\, \mathrm{trace}(\omega \cdot T)} \qquad (\omega = \omega^t \in \mathbb{C}^{2\times 2} \text{ has } \dots)'$$

$$= \sum_{x,y \in L} e^{2\pi i\left(\tau \frac{x^2}{2} + z\, x\cdot y + \tau' \frac{y^2}{2}\right)} \qquad \left(\omega = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}\right)$$

<u>Theorem</u> L even, $L = L^\vee$, $\subseteq \mathbb{R}^n$, n even. ~~$n \equiv 0 \dots$~~

There exists $\Gamma \subseteq Sp_2(\mathbb{R}) = \{\alpha \in Gl_4(\mathbb{R}) \mid \alpha^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\}$

such that

$$\vartheta_L^{(2)}\left((A\omega + B)(C\omega + D)^{-1}\right) \left|C\omega + D\right|^{-n/2} = \vartheta_L^{(2)}(\omega)$$

for all $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \Gamma$. (i.e. $\vartheta_L^{(2)}$ is a <u>Siegel modular form</u> of degree two).