(5)

~~**Theorem** If $f(x,y)$ has "complex multiplication", then $\sum a(n)$~~

**Theorem** Under suitable assumptions about $f(x,y)$ there exists $N \in \mathbb{Z}^{>0}$ (dep. on $f$) such that

$$\sum_{n\geq 1}^{\infty} a(n)\, q^n \in M_2(\Gamma_0(N)) \quad \left(\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; N \mid c \right\}\right)$$

**Conjecture** (Weil–Taniyama) True without any assumpt. about $f(x,y)$.

Why is it interesting to know that a generating function is a modular form. Simply because the space of modular forms are finite dimensional. So if you construct enough functions which all lie in one and the same space of mod. forms then there must be identities between these function, i.e. between their coefficients. This leads to striking and often deep results. Here are two simpler examples

One knows

| $2k$ | $\dim M_{2k}(SL_2(\mathbb{Z}))$ | basis |
|---|---|---|
| 4 | 1 | $E_4$ |
| 6 | 1 | $E_6$ |
| 8 | 1 | $E_8$ |
| 10 | 1 | $E_{10}$ |
| 12 | 2 | $E_{12}$, $E_4^3$ |

**Corollary**: $E_4^2 = E_8$, $E_4 E_6 = E_{10}$ etc.

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\, \sigma_3(n-m)$$

$$11\, \sigma_9(n) = 21\, \sigma_5(n) - 10\, \sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(m)\, \sigma_5(n-m)$$