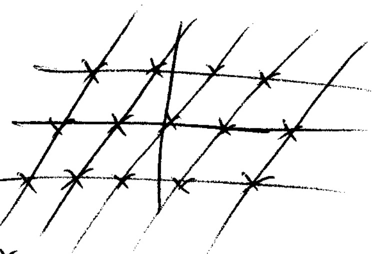
(2)

elliptic modular functions.

Ex 1 Let L = lattice in $\mathbb{R}^n$



Assume: L even, integral (i.e. $x\cdot y\in\mathbb{Z}$, $x\cdot x\in 2\mathbb{Z}$ $\forall x,y\in L$; "$\cdot$" usual scalar product in $\mathbb{R}^n$)

Set

$r_L(n)$ = # points in L lying on the sphere of radius $\sqrt{2n}$ $= \#\{x\in L \mid x^2 = 2n\}$

Clearly this is a diophantine question!

ex. $L = \mathbb{Z}^2$: $r_L(n) = \#\{(x_1,x_2)\in\mathbb{Z}^2 \mid x_1^2+x_2^2 = 2n\}$

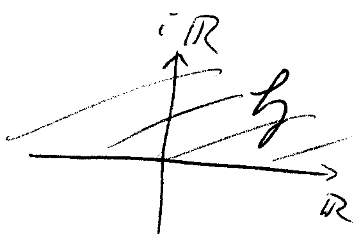
$L = \sqrt{2}\,\mathbb{Z}$: $r_L(n) = \#\{x\in\mathbb{Z} \mid x^2 = n\} = \begin{cases} 0 & \text{otherwise} \\ 1 & \text{if } n = \square \end{cases}$

$L = \sqrt{2}\,(\mathbb{Z}e^{2\pi i/6}+\mathbb{Z})$: $r_L(n) = \#\{(x_1,x_2)\in\mathbb{Z}^2 \mid x_1^2+x_1x_2+x_2^2 = n\}$.

Generating function

$$\vartheta_L := \sum_{n=0}^{\infty} r_L(n)\, q^n = \sum_{x\in L} q^{x^2/2}.$$

Put $q = e^{2\pi i\tau}$, $\tau\in\mathfrak{H}$, $\mathfrak{H} = \{\tau\in\mathbb{C} \mid \operatorname{Im}\tau > 0\}$



Then ϑ_L becomes a function in τ: $\vartheta_L = \vartheta_L(\tau)$ and has then very special properties

(i) $\vartheta_L(\tau+1) = \vartheta_L(\tau)$ ($\tau+1\to\tau$ leaves $q = e^{2\pi i\tau}$ invariant)

(ii) $\vartheta_L(\frac{-1}{\tau})\,\tau^{-n/2} = c(L)\,\vartheta_{L^*}(\tau)$ with a const $c(L)$ (dep. only on L)

where L^* = dual of L $= \{y\in\mathbb{R}^n \mid x\cdot L\subseteq\mathbb{Z}\}$