(1) Rewrite $\phi \overset{\simeq}{\Longleftrightarrow} (h_1, \dots, h_{2m})$ as

$$J_{k,m} \xrightarrow{\simeq} \mathrm{Hom}_{\tilde{\Gamma}}(\nabla h_m^*, M_{k-1/2})$$
$$\left(\phi \longmapsto \varphi, \quad \varphi(\lambda) = \sum_{\rho=1}^{2m} h_\rho \, \lambda(\vartheta_{m,\rho})\right)$$

To see what the spaces $M_{k-1/2}(\Gamma_0^0(n,n'),\chi)$ should have to do with $J_{k,m}$, use

(2) $M_{k-1/2}(\Gamma_0^0(n,n'),\chi) \xrightarrow{\simeq} \mathrm{Hom}_{\widetilde{\Gamma_0^0(n,n')}}(V_\kappa, M_{k-1/2}) \underset{\text{(universal property)}}{\xrightarrow{\simeq}} \mathrm{Hom}_{\tilde{\Gamma}}(V_\kappa^{\tilde{\Gamma}}, M_{k-1/2})$
$h \longmapsto (v_0 \to h)$

where κ is the linear character of $\widetilde{\Gamma_0^0(n,n')}$ determined by
$$h|\alpha = \kappa(\alpha)\, h \quad \text{for all } h \in M_{k-1/2}(\Gamma_0^0(n,n'),\chi),\ \alpha \in \widetilde{\Gamma_0^0(n,n')}$$
and $\mathbb{C}_\chi = \mathbb{C} v_0$ a one-dimensional $\widetilde{\Gamma_0^0(n,n')}$-module with character κ,
$V_\kappa^{\tilde{\Gamma}}$ is the induced $\tilde{\Gamma}$-module

What one ~~searches for~~ want to have ~~is an embedding~~
$$\cancel{\mathrm{Hom}_{\tilde{\Gamma}}(\nabla h_m^*, M_{k-1/2}) \hookrightarrow \mathrm{Hom}_{\tilde{\Gamma}}}(V_\kappa^{\tilde{\Gamma}}, M_{k-1/2})$$
~~induced from a $\tilde{\Gamma}$-embedding~~
is
$$\cancel{\nabla h_m^* \hookrightarrow V_\kappa^{\tilde{\Gamma}}}.$$
$$V_\kappa^{\tilde{\Gamma}} \simeq Th_m^* \oplus \text{Complement (as } \tilde{\Gamma}\text{-modules)},$$
so that
$$\mathrm{Hom}_{\tilde{\Gamma}}(V_\kappa^{\tilde{\Gamma}}, M_{k-1/2}) \simeq \mathrm{Hom}_{\tilde{\Gamma}}(\nabla h_m^*, M_{k-1/2}) \oplus \mathrm{Hom}_{\tilde{\Gamma}}(\text{Complement}, M_{k-1/2}).$$

Now
$$\nabla h_m^* \hookrightarrow V_\kappa^{\tilde{\Gamma}} \quad (\text{as } \tilde{\Gamma}\text{-modules})$$
does not work in general as a careful analysis of the corresponding $\tilde{\Gamma}$-modules shows.
So instead

(3) Decompose ∇h_m^* in irreducible $\tilde{\Gamma}$-modules, i.e. decompose ~~∇h_m~~
$$\nabla h_m = \bigoplus_i \nabla h_m^{(i)} \quad (\text{as } \tilde{\Gamma}\text{-module}).$$
As it will turn out, Th_m is multiplicity free, so that this decomposition is canonical.

(4) Derive from this a (canonical) decomposition
$$J_{k,m} = \bigoplus J_{k,m}^{(i)}$$
where
$$J_{k,m}^{(i)} \simeq \mathrm{Hom}_{\tilde{\Gamma}}((\nabla h_m^{(i)})^*, M_{k-1/2}).$$