It is not hard to show that the spaces $J_{k,m}$ are finite dimensional, $J_{k,m} = \{0\}$ for $k \leqq 0$ ~~or~~ $m \leq 0$ ~~or~~ $(k,m) = (0, \neq 0)$, $J_{k,0} \cong M_k(\Gamma)$. So I shall assume from now on that

$$k, m > 0.$$

The starting point for the connection between Jacobi-forms and modular forms of half integral weight is the easily proved fact that each $\phi \in J_{k,m}$ may be written as

(1) $$\phi = \sum_{\rho=1}^{2m} h_\rho(\tau)\, \vartheta_{m,\rho}(\tau, z)$$

where

$$\vartheta_{m,\rho} = \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \rho \bmod 2m}} q^{r^2/4m} \zeta^r$$

and the $h_\rho(\tau)$ are certain holomorphic functions on the upper half plane. They are uniquely determined by ϕ because the $\vartheta_{m,\rho}$ $(\rho = 1, \dots, 2m)$ are – considered as function in z for fixed τ – are linearly independent.

Namely, if ϕ

$$\phi = \sum c(n,r)\, q^n \zeta^r,$$

then there is a sequence $c_\rho(N)$, $\rho \in \mathbb{Z}$, $N \geq 0$ such that

$$c(n,r) = c_r(4mn - r^2)$$

This is an easy consequence of $\phi|_{k,m} x = \phi$ for $x \in \mathbb{Z}^2$. ~~and~~ Setting

$$h_\rho = \sum_{\substack{N \geq 0 \\ N \equiv -\rho^2 \bmod 4m}} c_\rho(N)\, q^{N/4m}$$

it is easy to verify (1). Note that the condition ii) above is exactly what is needed to have no negative powers of q in the Fourier-development of the h_ρ.

Now Γ acts on $Th_m := \mathrm{Span}\langle \vartheta_{m,\rho} \mid \rho \in 1, \dots, 2m \rangle$ by $(\vartheta, A) \to \vartheta|_{1/2,m} A$ as is well known. This is not quite true; ~~because~~ this does not exactly define an action because of the ambiguity in the sign by the determination of $(c\tau + d)^{1/2}$. So one has to take the well known two-fold extension $\tilde{\Gamma} = \{(A, w(\tau)) \mid A \in \Gamma,\ w(\tau)^2 = c\tau + d\}$ and to define $(\vartheta, \alpha) \to \vartheta|_{1/2,m}\alpha$ (Definition "$|_{1/2,m}\alpha$" similar ~~defined~~ to "$|_{k,m} A$".) So we have a representation

$$R : \tilde{\Gamma} \to GL_{2m}(\mathbb{C})$$

such that

$$\begin{pmatrix} \vartheta_{m,1} \\ \vdots \\ \vartheta_{m,2m} \end{pmatrix} \Big|_{1/2,m} \alpha = R(\alpha) \begin{pmatrix} \vartheta_{m,1} \\ \vdots \\ \vartheta_{m,2m} \end{pmatrix} \qquad (\forall \alpha \in \tilde{\Gamma}).$$