On the ~~connection~~ connection between Jacobi-forms and modular forms of half-integral weight

~~The title of this talk is~~

I shall talk here about the connection between Jacobi-forms and modular forms of half-integral weight. To be more precise, by "Jacobi-form" I always mean a Jacobi-form on the full modular group $Sl_2(\mathbb{Z})$. But before going into details let me first recall some basic facts and definitions.

I use Γ for the group $Sl_2(\mathbb{Z})$, Γ^J for $\Gamma \ltimes \mathbb{Z}^2$, i.e. $\Gamma^J = \{(A,x) \mid A \in \Gamma, x \in \mathbb{Z}^2\}$, with the composition law

$$(A,x)(A',x') = (AA', xA'+x').$$

I always ~~view~~ view Γ as a subgroup of Γ^J:

$$\Gamma \hookrightarrow \Gamma^J .$$

~~The group Γ^J acts on~~

For each pair of integers k,m there is an action of Γ^J on the space of functions $\phi(\tau,z)$ on $\mathfrak{H} \times \mathbb{C}$, where $\mathfrak{H}$ denotes the upper half plane. Namely with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$(\phi|_{k,m}(A,(\lambda,\mu)))(\tau,z) = \frac{e^m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z\right)}{(c\tau+d)^k}\,\phi\left(A\tau, \frac{z+\lambda\tau+\mu}{c\tau+d}\right),$$

where $e^m(\ldots) = e^{2\pi i m(\ldots)}$

I assume that most of you are acquainted with this strange looking action. ~~For~~ $\mu=0$ this is similar to the usual action of Γ ~~on~~ functions in the upper half plane, for A = identity this corresponds — for fixed τ — to the transformation law for certain theta series on the elliptic curve $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$.

I use $J_{k,m}$ for the space of Jacobi-forms on Γ of ~~weight~~ weight k, index m, i.e.

$$J_{k,m} = \left\{ \phi : \mathfrak{H}\times\mathbb{C} \xrightarrow{\text{holomorphic}} \mathbb{C} \;\middle|\; \begin{array}{l} i)\ \phi|_{k,m}\gamma = \phi \quad \forall \gamma \in \Gamma^J \\ ii)\ \phi = \sum\limits_{4mn-r^2 \geq 0} c(n,r)\, q^n \zeta^r \end{array} \right\},$$ where $q = e(\tau)$, $\zeta = e(z)$.

That a ϕ fullfilling i) has a Fourier development in terms of q and ζ is clear, so the essential thing in ii) is the extra condition "$4mn-r^2 \geq 0$". ~~This condition is exactly what one needs to deduce the regularity in the cusp for the modular forms of half-integral weight that I shall ass~~ ~~It turns out that~~ I shall come back to this in a moment.