4. Beispiele

Sei p Primzahl, $p \neq 2$, $\zeta = e^{2\pi i/p}$, $k = \mathbb{Q}(\zeta + \zeta^{-1})$.

$V := \mathbb{Q}(\zeta)$, Skalarprodukt auf V: $v \cdot w := \frac{v \cdot \bar{w} + \bar{v} \cdot w}{p}$

$\Gamma = (1-\zeta)$.

~~Γ ganz und gerade~~:

Arithmetik in V, k:

$(p) = \cancel{(1-\zeta)} \left(\frac{1+\dots+1}{p\text{-mal}}\right) = \left(\prod_{i=1}^{p-1}(1-\zeta^i)\right) = \prod_i (1-\zeta^i) = (1-\zeta)^{p-1}$ $\left(\frac{1-\zeta^i}{1-\zeta} = \text{Einheit}\right)$

$(p) = \left((1-\zeta)(1-\bar{\zeta})\right)^{\frac{p-1}{2}}$ in k, insbesondere $n_{V|\mathbb{Q}}((1-\zeta)(1-\bar{\zeta})) = p$ ($n(\cdot)$ ist multiplikativ)

$\mathcal{D}_V = (1-\zeta)^{p-2}$ $\left(n_{V|\mathbb{Q}}(\mathcal{D}_V) = |D_V| \text{ Diskriminante von } V := \det(\operatorname{tr} e_i e_j)^2\right)$,

wo $\mathcal{O}_V = \mathbb{Z}e_1 + \dots + \mathbb{Z}e_{p-1}$; also $\mathcal{O}_V = \mathbb{Z} + \mathbb{Z}\zeta + \dots + \mathbb{Z}\zeta^{p-2}$, also

$D_V = \det\begin{pmatrix} 1 & \zeta & \dots & \zeta^{p-2} \\ 1^2 & \zeta^2 & \dots & (\zeta^{p-2})^2 \\ 1^{p-1} & \zeta^{p-1} & & (\zeta^{p-2})^{p-1}\end{pmatrix}^2 = \pm\prod_{\substack{i,j=1 \\ i\neq j}}^{p-1}(\zeta^i - \zeta^j) = \prod_i \left(\zeta^i\left(\prod_{j\neq i}(1-\zeta^{j-i})\right)\right) = \left(\prod_i \frac{\zeta^i}{1-\zeta^i}\right) p^{p-1} = \pm p^{p-2}$

$\mathcal{D}_k = \left((1-\zeta)(1-\bar{\zeta})\right)^{\frac{p-3}{2}}$ (Ist $\mathfrak{p}$ Primideal in k, $\mathfrak{P}^e | p\ (\in \mathbb{Z})$, so $\mathfrak{p}^{e-1} | \mathcal{D}_k$,

also $\left((1-\zeta)(1-\bar{\zeta})\right)^{\frac{p-3}{2}} | \mathcal{D}_k$. Ferner $\mathcal{D}_{V|\mathbb{Q}} = \mathcal{D}_{V|k}\,\mathcal{D}_{k|\mathbb{Q}}$, wie oben folgt $\cancel{(1-\zeta)(1-\bar{\zeta})} | \mathcal{D}_{V|k}$)

also $\mathcal{D}_{k|\mathbb{Q}} = \frac{\mathcal{D}_{V|\mathbb{Q}}}{\mathcal{D}_{V|k}} \,\Big|\, \frac{(1-\zeta)^{p-2}}{(1-\zeta)^2} = (1-\zeta)^{p-3} = \left((1-\zeta)(1-\bar{\zeta})\right)^{\frac{p-3}{2}}$)

Damit haben wir:

Γ ganz und gerade: $v, w \in \Gamma \Leftrightarrow v, w \equiv 0\ ((1-\zeta)) \Rightarrow \operatorname{tr}_{V|\mathbb{Q}} v\cdot\bar{w} \equiv 0 \bmod (1-\zeta)$,

aber $p \in 1-\zeta$, $\operatorname{tr} v \cdot w \in \mathbb{Z}$, also $p | \operatorname{tr} v\cdot w$.

~~$\operatorname{tr}_{k|\mathbb{Q}} \mathcal{O}_k(v\bar{w} + \bar{v}w) \subseteq \operatorname{tr}_{V|\mathbb{Q}} \mathcal{O}_V v\bar{w} \in 2\mathbb{Z}$, da $\operatorname{Re} v\bar{w} \in \mathcal{O}_k$~~

~~$\subseteq \mathbb{Z}\operatorname{tr}_{V|\mathbb{Q}} \zeta + \mathbb{Z}\operatorname{tr}_{V|\mathbb{Q}} \zeta^2 + \dots \subseteq \mathbb{Z}(1+\zeta+\zeta^2+\dots+\zeta^{p-1})$~~

$\subseteq \cancel{\mathbb{Z}}\ \mathbb{Z}\operatorname{tr} \zeta^i \zeta$

$\operatorname{tr}_{k|\mathbb{Q}} \mathcal{O}_k(v\cdot\bar{v} + \bar{v}\cdot v) = 2\operatorname{tr}_{k|\mathbb{Q}} \mathcal{O}_k v\cdot\bar{v} \in 2\mathbb{Z}$.