From this we now obtain

$$\sum_{n=1}^{\infty} \sum_{q\geq 1} (-1)^{q+1} d_{n,q}(X) \frac{T^n}{n+1} = \sum_{n=0}^{\infty} \binom{X+n}{n} T^n - \sum_{n=0}^{\infty} \frac{T^n}{n+1}$$

$$= \sum_{n=0}^{\infty} \binom{-X-1}{n} (-T)^n + \frac{\log(1-T)}{T}$$

$$= \exp(-X \log(1-T)) \cdot \frac{1}{1-T} + \frac{\log(1-T)}{T}$$

Hence

$$\sum \frac{\zeta_{\Delta,n}(0)}{n+1} T^n = \text{last right hand side with } X^r \mapsto \zeta(-r)$$

$$= \left(\sum_{r=0}^{\infty} \frac{\zeta(-r)}{r!} (-1)^r \log(1-T)^r \right) \cdot \frac{1}{1-T} + \zeta(0) \frac{\log(1-T)}{T}$$

But

$$\sum_{r=0}^{\infty} \frac{\zeta(-r)}{r!} u^r = -\sum \frac{B_{r+1}(1)}{(r+1)!} u^r = -\frac{e^u}{e^u - 1} + \frac{1}{u}$$

and with $u = \log \frac{1}{1-T}$:

$$= -\frac{\frac{1}{1-T}}{\frac{1}{1-T} - 1} - \frac{1}{\log(1-T)}$$

Thus we conclude

Prop

$$\sum_{n=0}^{\infty} \frac{\zeta_{\Delta,n}(0)}{n+1} T^n = -\frac{1}{(1-T)T} - \frac{1}{(1-T)\log(1-T)} - \frac{1}{2} \frac{\log(1-T)}{T}$$

n	1	2	3	4	11
$\zeta_{\Delta,n}(0)$	$-\frac{2}{3}$	$-\frac{11}{8}$	$-\frac{379}{180}$	$-\frac{821}{288}$	$-\frac{180277161763}{217945728000}$