proof of the Lemma:

denominate from the defining equation of the $d_{n,q}(x)$.

(e.g. $\binom{X-(n+1-q)+n}{n} = \binom{-X+q-1}{n} = (-1)^n \binom{X+n-q}{n}$ etc.)

Thus we have

$$\varphi_{\Delta,m}(\omega) = \sum (-1)^{q+1} d_{n,q}(X)\Big|_{X^r \to \beta(-r)}$$

Here we use

Lemma (cf. Lemma 2.4.1 / Zagier)

$$\sum (-1)^{q+1} d_{n,q}(X) = (n+1)\binom{X+n}{n} - 1 \qquad \text{for } n \geq 0$$

proof: can be checked for $n=0$.