proof

Write for $\mathrm{Re}\, s \gg 0$

$$Z(s) = \sum_{1\le n\le a}' \frac{P(n)}{n^s(n+a)^s} + \sum_{n>a}' \frac{P(n)}{n^{2s}\left(1+\frac{a}{n}\right)^s}$$

$$= \quad '' \quad + \sum_{n>a}' \frac{P(n)}{n^{2s}} \sum_{l=0}^{\infty} \binom{-s}{l}\left(\frac{a}{n}\right)^l$$

$$= \quad '' \quad + \sum_{r} c_r \sum_{l=0}^{\infty} \binom{-s}{l} a^l \sum_{n>a}' \frac{1}{n^{2s+l-r}}$$

$$= \quad '' \quad + \sum_{r} c_r \sum_{l=0}^{\infty} \binom{-s}{l} a^l \left\{ \zeta(2s+l-r) - \sum_{1\le n\le a}' \frac{1}{n^{2s+l-r}} \right\}$$

~~where $\zeta(s) =$~~

Using

$$\zeta(2s+l-r) - \sum_{1\le n\le a}' \frac{1}{n^{2s+l-r}} = O\left(\frac{1}{(a+\varepsilon)^l}\right) \quad \text{for } l\to\infty,$$

uniformly in all regions $\mathrm{Re}\, s >$ const.

and for any $\varepsilon' > 0$

$$\binom{-s}{l} = O\left(\frac{1}{\varepsilon'^l}\right), \text{ uniformly on compact subsets of } \mathbb{C}$$

(proof e.g. by $\binom{-s}{l} = \frac{1}{2\pi i} \int_{|t|=\varepsilon'} \frac{(1+t)^{-s}}{t^{l+1}}\, dt$

we see that the right hand side of the above identity is normally convergent on $\mathbb{C}$.

This gives the meromorphic continuation

To investigate the behaviour of $Z(s)$ near $s=0$ note that in right hand side of the above identity for $Z(s)$ we have:

$$\binom{-s}{l} = 0 \quad \text{for } s=0,\ l\ge 1$$

$\zeta(2s+l-r)$ defined at $s=0$ for all l, r except for $l-r=1$