Then we have to compute $\sum_n \zeta_{D,n}\frac{(0)}{n+1} T^n \in \mathbb{C}[[X]]$,

and for that we have to investigate more closely $\zeta_q(s)$.

Lemma 1

$d_{n,q}(X)$ is a polynomial in X, we have

$$d_{n,q}(1) = d_{n,q}(2) = \dots = d_{n,q}(\cancel{q-1}) = 0$$

proof $X \,\Big|\, \binom{X+n-q}{n} = \frac{(X+n-q)(X+n-q-1)\cdots(X-q+1)}{n!}$

$(X+n+1-q) \,\Big|\, \binom{X+n}{n}$

and

$\binom{X+n-q}{n} = 0$ for $X = 1, 2, \dots, q-1$. $\square$

Hence

$$q\zeta_q(s) = \cancel{q}\sum_{k=1}^{\infty} \frac{d_{n,q}(k)}{k^s(k+n+1-q)^s}$$

and we are in the following situation.

Proposition Let $P(x) \overset{= \sum c_r X^r}{\in} \mathbb{C}[X]$, $a > 0$. Then

$$Z(s) := \sum_{k=1}^{\infty} \frac{P(k)}{k^s(k+a)^s}$$

converges $\overset{\text{absolutely and uniformly}}{\text{for}}$ $\operatorname{Re} s \gg 0$, can be meromorphically continued to $\mathbb{C}$, is holomorphic at 0 and

$$Z(0) = \sum_r c_r\, \zeta(-r) - \frac{1}{2}\int_{-a}^{0} P(x)\,dx ,$$

where

$\zeta(s) =$ ~~R-ex~~ Riemann zeta function $= \sum_1^{\infty} \frac{1}{n^s}$.

Recall

$$\zeta(-r) = -\frac{B_{r+1}(1)}{r+1} \qquad r = 0, 1, \dots$$

where $\frac{x e^{tx}}{e^x - 1} = \sum_r \frac{B_r(t)\, x^r}{r!}$