Starting point:

(*) $$\log 2 \cdot \sum (-1)^{q+1} q\, \zeta_q(0) = \begin{cases} \text{coefficient of } x^n \text{ in} \\ (n+1)\left(\dfrac{x}{1-e^{-x}}\right)^{n+1} F(x) \end{cases}$$

where

$$\zeta_q(s) = \frac{1}{q}\sum_{k\geq q} \frac{d_{n,q}(k)}{k^s(k+n+1-q)^s}, \quad (\text{cf. } (21)) \text{ Ikeda Taniguchi}$$

$$d_{n,q}(X) = q\binom{n}{q}\binom{X+n}{n}\binom{X+n-q}{n}\left[\frac{1}{X} + \frac{1}{X+n+1-q}\right]$$

(cf. (22))

Set

$$\zeta_{\Delta,n}(s) := \sum (-1)^{q+1} q\, \zeta_q(s).$$

~~Well first of all~~ we prove first of all that (*) makes sense, i.e. that there is ~~a~~ $F(x) \in \mathbb{C}[[x]]$ satisfying (*) and that it is unique. Namely, we have

Prop. There is one and only one F satisfying (*); one has

$$F(x) = e^{-x} \cdot \sum \frac{\zeta_{\Delta,n}(0)}{n+1}\left(1-e^{-x}\right)^n \cdot \log 2$$

proof

(*) can be written as

$$\frac{\log 2 \cdot \zeta_{\Delta,n}(0)}{n+1} = \operatorname{Res}_{x=0} \frac{F(x)\,dx}{(1-e^{-x})^{n+1}}$$

Set

$$1-e^{-x} = y$$

i.e. $-\ln(1-y) = x$, $\quad dx = \dfrac{dy}{1-y}$

then

$$= \operatorname{Res}_{y=0} \frac{F(x)\,dy}{y^{n+1}(1-y)}$$

$= n$-th coefficient of y^n in $\dfrac{F(x)}{1-y}$. ☐