(Product Formulas). *Let* $c = -\frac{444}{11}, -\frac{1420}{17}, -\frac{3164}{23}$, *and let* l *denote its denominator. For any integer* x, *relatively prime to* l, *set*

$$[x]_l = q^{l/24-(l-2x_0)^2/8l} \prod_{\substack{n \geq 1 \\ n \equiv \pm x \bmod l}} (1-q^n)^{-1},$$

where x_0 *is that integer which satisfies* $1 \leq x_0 \leq (l-1)/2$, $x_0 \equiv \pm x \bmod l$. *Then the set of functions* $\xi_{c,h}$ $(h \in H_c)$ *equals*

$$\begin{cases} \{\frac{\eta(q^l)}{\eta(q)} \frac{[4x]_l[5x]_l}{[x]_l}, \frac{\eta(q^l)}{\eta(q)} [x]_l\}_{x=1,\ldots,(l-1)/2} & \text{if} \quad l = 11 \\ \{\frac{\eta(q^l)}{\eta(q)} \frac{[6x]_l[7x]_l[8x]_l}{[x]_l}, \frac{\eta(q^l)}{\eta(q)} [x]_l[5x]_l\}_{x=1,\ldots,(l-1)/2} & \text{if} \quad l = 17 \\ \{\frac{\eta(q^l)}{\eta(q)} \frac{[8x]_l[9x]_l[10x]_l[11x]_l}{[x]_l}, \frac{\eta(q^l)}{\eta(q)} [x]_l[5x]_l[7x]_l\}_{x=1,\ldots,(l-1)/2} & \text{if} \quad l = 23. \end{cases}$$

Table 3: The functions $\theta(\tau; I, p) = \sum_{c \in I} \Phi(c)\, p(c)\, q^{n(c)/n(I)l}$

c	I	p
$-\frac{8}{5}$	$\mathfrak{O}$	$(\alpha+\overline{\alpha})(\beta-\overline{\beta})$
$\frac{4}{5}$	$\mathfrak{O}$	$-77760(\alpha^4+\overline{\alpha}^4)(\beta^4+\overline{\beta}^4)$ $+5508\sqrt{-5}\,(\alpha^4+\overline{\alpha}^4)(\beta^4-\overline{\beta}^4)$ $5\sqrt{-5}\,(\beta^4+\overline{\beta}^4)(\beta^4-\overline{\beta}^4)$
$-\frac{444}{11}$	$\mathfrak{O}$	$(-\alpha\overline{\alpha}+\frac{1}{3}\beta\overline{\beta})(\alpha+\overline{\alpha})(\beta\frac{1-\sqrt{-l}}{2}+\overline{\beta}\frac{1+\sqrt{-l}}{2})$
$-\frac{1420}{17}$	$\mathfrak{O}\mathfrak{p}$	1
$-\frac{3164}{23}$	$\mathfrak{O}$	$\sum_{j=0}^{7}\sqrt{-3}^{\epsilon_j}(\alpha^j+\overline{\alpha}^j)(\overline{\gamma_j}\beta^{8-j}+\gamma_j\overline{\beta}^{8-j})$

— **A quaternion algebra:** $V = K + Ku$, where $K = \mathbb{Q}(\sqrt{-l})$, $u^2 = -1/3$ and $\alpha u = u\overline{\alpha}$ for all $\alpha \in K$.

— **A Maximal order in** V**:** $\mathfrak{O} = \mathfrak{o} + \mathfrak{p}v$ with $v = u$ for $l \equiv 3 \bmod 4$, and $v = \frac{1+u}{2}$ for $l \equiv 1 \bmod 4$, where $\mathfrak{o}$ is the ring of integers in K, and $\mathfrak{p}$ a prime ideal in $\mathbb{Q}(\sqrt{-l})$ such that $3 = \mathfrak{p}\overline{\mathfrak{p}}$.

— **Further ingredients:** $\Phi: \mathfrak{O} \to \mathbb{C}^{l-1}$, r-th components of $\Phi(c) = \chi(c)$ if $n(c) \equiv r \bmod l$, $= 0$ otherwise, with a fixed character χ of order 3 on $\mathbb{F}_{l^2} = \mathfrak{O}/\sqrt{-l}\mathfrak{O}$, and p homogeneous polynomial function on V which is spherical w.r.t. to the reduced norm $n(c) = |\alpha|^2 + \frac{1}{3}|\beta^2|^2$ $(c = \alpha + \beta u)$.