Theorem [~]

$k \in \frac{1}{2}\mathbb{Z}$, $\rho : \tilde{\Gamma} \longrightarrow GL(n, \mathbb{C})$ rep. with finite image and

$\rho((\pm id, \varepsilon)) = \bar{\varepsilon}^{2k}\, id$ for all $(\pm id, \varepsilon) \in \tilde{\Gamma}$.

Then

$$\dim M_k(\rho) - \dim S_{2-k}(\bar{\rho}) = \frac{k-1}{12}\cdot n + \frac{1}{4}\,\mathrm{Re}\left(e^{\pi i k/2}\, \mathrm{tr}\, \rho(S, \sqrt{\tau})\right) + \frac{2}{3\sqrt{3}}\,\mathrm{Re}\left(e^{\pi i(2k+1)/6}\, \mathrm{tr}\, \rho(ST, \sqrt{\tau+1})\right) + \frac{1}{2}\, u(\rho) - \sum_{j=1}^{n} \mathbb{B}_1(\lambda_j).$$

Here $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

λ_j $(1 \le j \le n)$ s.th. $e^{2\pi i \lambda_j}$ runs thr. eigenval. of $\rho(T)$,

$u(\rho) = \#\, j$ s.th. $e^{2\pi i \lambda_j} = 1$

$\mathbb{B}_1(x) = x' - \frac{1}{2}$ if $x \in x' + \mathbb{Z}$, $0 < x' < 1$, $= 0$ ~~[illegible]~~ if $x \in \mathbb{Z}$.

$S_{..}(\rho)$ def. like $M_{..}(\rho)$ but with big O repl. by small o.

How to obtain a basis for $M_k(\rho)$?

- ~~Answer~~ useful for
  - obtaining explicit formulas for conf. char.
  - if dim-comput. do not suffice to conclude uniqueness

Proof: One has $\mathbb{C}(\rho) = \cancel{\mathrm{coll.}}\,\mathbb{C}^n$ with $\tilde{\Gamma}$-right action $(z, \alpha) \longmapsto \rho(\alpha)^t z$.

$M_k(\rho) \ni F \longmapsto \left(\mathbb{C}(\rho) \ni z \longmapsto z^t \cdot F \in M_k(\ker \rho)\right) \in \mathrm{Hom}_{\tilde{\Gamma}}(\mathbb{C}(\rho), M_k(\ker\rho))$

is isom.

$$\dim \mathrm{Hom}_{\tilde{\Gamma}}(\mathbb{C}(\rho), -) = \frac{1}{[\tilde{\Gamma} : \ker\rho]} \sum_{\alpha \in \tilde{\Gamma}/\ker\rho} \overline{\mathrm{tr}\,\rho(\alpha)}\; \mathrm{tr}(\alpha, M_k(\ker\rho))$$

Apply Eichler-Selberg trace formula. $\square$