To identify ϱ (or $\eta^k \otimes \varrho$):

$T = \begin{pmatrix}1&1\\0&1\end{pmatrix}$, $\varrho(T)$ = diag. matrix w. roots of 1 on diag., $\varrho(T)^N = \mathrm{id}$ some N

Ker ϱ ~~congr.~~ subgroup.

<u>Theorem</u> [Fricke-Klein-Wohlfart]

$\varrho : \Gamma \to GL(n,\mathbb{C})$ any rep., Ker ϱ = congruence subgroup.

Let $N \in \mathbb{Z}_{>0}$ s.t. $\varrho(T)^N = \mathrm{id}$.

Then Ker $\varrho \supseteq \Gamma(N)$.

Thus ϱ factors to $\underline{\varrho} : \Gamma/\Gamma(N) = SL(2, \mathbb{Z}/N\mathbb{Z}) \longrightarrow GL(n,\mathbb{C})$.

Only finitely poss. for ϱ. To further reduce possibilities:

$\eta^k \xi$ has rat. Fourier coeff.

~~Theorem~~ <u>Corollary</u>

$\varrho : \Gamma \to GL(n,\mathbb{C})$ ~~rep.~~, Ker $\varrho \supseteq \Gamma(N)$ some $N \in \mathbb{Z}_{>0}$, let $K = \mathbb{Q}(e^{2\pi i/N})$.

~~If then~~ Assume there ex. $F \in M_k(\varrho)$ (some k) such that

F has Four. coeff. in K.

Then $\varrho(\Gamma) \subseteq GL(n,K)$.

Using this ϱ was reduced in all cases to one or two poss.

To see how much freedom is left one has to look at the dimension

of the spaces $M_k(\varrho)$.

<u>Theorem</u>: $k, N \in \mathbb{Z}$, $N > 0$, $K = \mathbb{Q}(e^{2\pi i/N})$. Then the space

$M_k^K(\Gamma(N))$ = mod. forms in $M_k(\Gamma(N))$ whose Fourier coeff.

(at ∞) are in K is invariant under $(f,A) \mapsto f|_k A$.