~~The basic properties are~~:

Here $\Gamma^*_{k,m} = \Gamma_{k,m}$ but with $(c\tau+d)^{+k}$ replaced by $|c\tau+d|(c\bar\tau+d)^{k-1}$.

The $J^*_{k,m}$ these skew-holomorphic Jac-forms have essentially the same basic features as the holomorphic one, except that here ~~the Fourier coefficients are essentially indexed by non negative discriminant~~ positive discriminants play the role of the negative discriminants in the theory of hol. Jac-forms; i.e. to any skew-hol. Jac-f. ϕ correspond a collection of coefficients $c_\phi(\Delta,r)$ with $\Delta \equiv r^2 (4m)$, $\Delta \geq 0$, depending only on

~~An example~~ an illustration: Example

(A) $J^*_{k,m}$ is a free module over $\bigoplus_k M_k(1)$ of rank 2 (via $(\phi, E) \mapsto \phi(\tau,z)\cdot E(-\bar\tau)$). Generators are

$$A = \sum_{x,y\in\mathbb{Z}} e^{\pi i\tau(2xy+2z(x+y))}\, e^{-2\pi v\frac{(x-y)^2}{2}} \in J^*_{1,1}$$

$$B = \partial_{\bar\tau} A + \frac{\pi i}{12} A\, E_2(-\bar\tau) \in J^*_{3,1}.$$

With these skew-holomorphic Jacobi forms the above mentioned gap is filled. One has

add 4

$J_{k,m} \oplus J^*_{k,m}$ is Hecke-equiv. isomorphic to $\mathfrak{M}_{2k-2}(m)$. More precisely:

Main-Theorem (i) Let D be a fund. discr., $s\in\mathbb{Z}$ such that $s^2 \equiv D\ (4m)$. Then

$$\phi \mapsto \sum_{\ell\geq 0}\Big(\sum_{a|\ell} a^{k-2}\Big(\frac{D}{a}\Big)\, c_\phi\Big(\frac{\ell^2}{a^2}D, \frac{\ell}{a}s\Big)\Big)\, q^\ell$$

defines a map $\mathcal{S}_{D,r}: J_{k,m} \oplus J^*_{k,m} \longrightarrow \mathfrak{M}_{2k-2}(m)$. It commutes with Hecke operators, maps $J^{(*)}_{k,m}$ into $\mathfrak{M}^{(\mp)}$ (ii) Some linear comb. of the $\mathcal{S}_{D,r}$ is an isomorphism.