<u>Theorem</u> Let V be a real vector space of even dimension $n<\infty$, "$\cdot$" a scalar product on V (not nec. pos. def.), $p(x)$ be a real function on V such that

(i) $p(x)\, e^{-\pi x^2} \in \mathcal{S}(V)$

(ii) $\left(-\frac{1}{4\pi}\Delta + E\right) p = \left(k-\frac{n}{2}\right) p$ for some $k\in\mathbb{Z}$.

Let $\Gamma \subseteq V$ be an even integral lattice, $Z_0 \in \Gamma$.

Then

$$\vartheta_{\Gamma,p}(\tau,z) := \sum_{Z\in\Gamma} v^{\frac{n}{4}-\frac{k}{2}}\, p\left(\sqrt{v}\left[Z+\frac{y}{v}Z_0\right]\right) e\left(\tau\frac{Z^2}{2} + z\, Z\cdot Z_0\right)$$

transforms like a Jacobi form of weight k, index $m = \frac{Z_0^2}{2}$ on $\Gamma_1(\ell)$ (ℓ = level of Γ).

Here we can find functions like we are looking for:

Let signature of $\cdot$ = $1, n-1$, $Z_0^2>0$

$p(Z) = q(Z_\perp)\, e^{2\pi Z_\perp^2}$ where $Z = \alpha Z_0 + Z_\perp$ = orthogonal decomp. of Z, $q(x)$ = spherical polynomial on $Z_0^\perp \subseteq V$ with respect to restr. of "$\cdot$" to $Z_0^\perp$

Then $(\operatorname{Im}\tau)^{-\frac{n}{4}} \times$ corresponding $\vartheta(\tau,z)$ ~~satisfies in an element of the~~ has exactly the properties listed below our definition of "skew-holomorphic Jacobi form":

$J^*_{k,m}$ = skew-hol. Jacobi forms of index m, weight k on $SL_2(\mathbb{Z})$

$$= \left\{ \phi : \mathfrak{H}\times\mathbb{C} \xrightarrow{\text{smooth}} \mathbb{C} \;\middle|\; \begin{array}{l} \text{(i) } \phi(\tau,z) \text{ is hol. in } z \text{ and } (8\pi i m\,\partial_\tau - \partial_z^2)\,\phi = 0 \\ \text{(ii) } \phi|^*_{k,m}\gamma = \phi \quad \forall \gamma\in\Gamma^J \\ \text{(iii) } \phi \text{ has a Fourier development of the form} \\ \quad \phi(\tau,z) = \sum_{\substack{n,r \\ r^2-4mn\geq 0}} c(n,r)\, q^n \zeta^r e^{-\pi\frac{r^2-4mn}{m}\operatorname{Im}\tau} \end{array} \right\}$$