(5)

Back to Siegel modular forms: Any Siegel modular form has an ~~Fourier~~ expansion in terms of Jacobi forms. Thus it lies at hand to try the following

Program: Study Siegel modular forms via their Fourier Jacobi expansion.

So far I know there are only two main results known.

Theorem (Maass ~~1979~~)

There exist Operators $V_m : J_{k,1} \longrightarrow J_{k,m}$ (given explicitly by $\sum c(n,r) q^n \zeta^r \longrightarrow \sum_{n,r} \sum_{a|(n,r,m)} a^{k-1} c(\frac{mn}{a^2}, \frac{r}{a}) q^n \zeta^r$) such that $\phi \mapsto \phi | V := \sum \phi|V_n \, e^{2\pi i n \tau'}$ defines a Hecke-equivariant embedding of $J_{k,1} \hookrightarrow M_k(Sp_2(\mathbb{Z}))$.

The image of this embedding is the so-called Maass-space. This Theorem is part of the proof of the Saito-Kurokawa-conjecture.

The second Theorem along the lines of the above program is:

Theorem (Kohnen / ~~S~~ 1988)

~~For~~ $F = \sum \varphi_n e^{2\pi i n \tau'}$, $G = \sum \psi_n e^{2\pi i n \tau'} \in M_k^{cusp}(Sp_2(\mathbb{Z}))$ set

$$D_{F,G}(s) = \zeta(2s-2k+4) \sum_{n=1}^{\infty} \frac{\langle \varphi_n | \psi_n \rangle}{n^s}.$$

(i) The $D_{F,G}(s)$ has a merom. contin. to $\mathbb{C}$ with possible a pole only at $s=k$ of order 1 and residue = (constant depending on $k$) $\cdot \langle F, G \rangle$. It satisfies $D^*_{F,G}(s) = (2\pi)^{-2s} \Gamma(s) \Gamma(s-k+2) D_{F,G}(s) = D^*_{F,G}(2k-2-s)$.

(ii) Let $k$ be ~~even~~. If $G \in M_k^{Maass}(Sp_2(\mathbb{Z}))$ and $F$ is a Hecke-Eigenform then

$$D_{F,G}(s) = \langle \varphi_1 | \psi_1 \rangle \cdot (\text{Spinor (or Andrianov)} - \text{Zeta function of } F).$$

The proof of (i) is essentially an adaption of the Rankin-Selberg method to Siegel modular forms. The ~~second~~ proof of (ii) uses Hecke theory for Jacobi forms and for Siegel mod. forms à la Andrianov.