$J(\mathbb{R})$ acts on $\mathfrak{H}\times\mathbb{C}$: $\gamma\cdot(\tau,z) = \left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)$ $\left(A=\begin{pmatrix}a&b\\c&d\end{pmatrix}, \gamma = [A,(\lambda,\mu),\zeta]\right)$,
and for each pair $k,m\in\mathbb{Z}$
it acts on functions on $\mathfrak{H}\times\mathbb{C}$:

$$(\phi|_{k,m}\gamma)(\tau,z) = \phi(\gamma\cdot(\tau,z))\,(c\tau+d)^{-k}\, e^{m}\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu\right)\zeta^{m}$$

Then define for integral k,m

$J_{k,m}$:= space of Jacobi forms of weight k and index m $= \left\{\phi:\mathfrak{H}\times\mathbb{C}\to\mathbb{C} \;\middle|\; \begin{array}{l} \text{i) } \phi \text{ holomorphic in } \tau \text{ and } z \\ \text{ii) } \phi|_{k,m}\gamma=\phi \text{ for all } \gamma\in SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2 \\ \text{iii) } \phi \text{ has a Fourier development of the form} \\ \quad \phi(\tau,z)=\sum\nolimits'_{\substack{r,n\in\mathbb{Z}\\ r^2-4mn\le 0}} c(n,r)\,q^n\zeta^r \end{array}\right\}$

(with $q=e^{2\pi i\tau}$, $\zeta=e^{2\pi i z}$)

For $m<0$ $J_{k,m}=0$ and $J_{k,0}$ = usual modular forms (norm. at ∞). Hence forth always $m>0$.

~~Here $SL_2(\mathbb{Z})^J$:=~~ Note that the condition "$r^2-4mn\le 0$" is exactly the regularity condition fullfilled by ϕ_m in the development of F. Note that $c(n,r)$ in ϕ_m is nothing else but the coefficient $A(n,r,m)$ in the usual Fourier-development of F, & viewing n,r,m as the coefficients of a binary quadratic form, it is well-known that $A(n,r,m)$ depends only on the $SL_2(\mathbb{Z})$-equivalence class of (n,r,m), thus $c(n,r)$ depends only on r^2-4mn, r mod $2m$. This turns out to be true for any Jacobi form:

(*) Fourier-development of a $\phi\in J_{k,m}$ can be written: $\phi = \sum_{\substack{\Delta\le 0,\, r\in\mathbb{Z}\\ \Delta\equiv r^2 (4m)}} C(\Delta,r)\, q^{\frac{r^2-\Delta}{4m}}\zeta^r$ with coefficient $C(\Delta,r)$ depending only on r modulo $2m$.

One could also define Jacobi forms on subgroups of $SL_2(\mathbb{Z})$ and also of $\frac{1}{2}$ integral weight. In this broader class of Jac forms the simplest ones are those which gave the whole theory its name, the

Jacobi theta functions: $\vartheta_{m,\rho}(\tau,z) = \sum_{\substack{r\in\mathbb{Z}\\ r\equiv\rho\,(2m)}} q^{\frac{r^2}{4m}}\zeta^r$ $(\rho=1,\dots,2m)$ $\left(\in J_{\frac{1}{2},m}(\Gamma(4m))\right)$

Revisiting them, we see:

any $\phi\in J_{k,m}$ can be written as $\phi = \sum_{\rho=1}^{2m} h_\rho(\tau)\,\vartheta_{m,\rho}(\tau,z)$ with suitable functions $h_\rho(\tau)$.