Proof. Two natural methods for producing congruences for theta series.

1/ $\Theta_L \equiv_{\ell} \Theta_{L^\sigma}$ $\quad \sigma \in \mathrm{Aut}(L)$ and $\sigma = \ell^t$ some t

2/ Lemma $\sigma \in \mathrm{Aut}(L)$, and $\sigma = \ell$, $s(L) \not\equiv 0 \bmod \ell$, $s(L^\sigma) \mid \ell^\infty$

Then $\Theta_L \mid_k A$ has ℓ-integral F.cfs $\quad (k = \frac{r}{2})$

and is $\equiv \Theta_L \bmod \ell$

Hence idea:

given ~~$s(L)$~~ L with $\ell / s(L) \mid \ell^n$.

- construct $\hat{L}$ s.t. there is $\sigma \in \mathrm{Aut}(\hat{L})$, and $\sigma = \ell^t$ (some $t > 0$) such that $L = \hat{L}^\sigma$ and $s(\hat{L}) \not\equiv 0 \bmod \ell$

- $\Theta_{\hat{L}} \mid A \equiv_{\ell} \Theta_{\hat{L}}$ implies $\Theta_{\hat{L}} \in M_k(\Gamma_0(\underbrace{s(\hat{L})}_{\hat{\ell}}))$. $\quad (k = \frac{\mathrm{rank}\,\hat{L}}{2})$

$$\underbrace{[Sl(2,\mathbb{Z}) : \Gamma_0(\hat{\ell})]}_{\hat{\ell} \prod_{p \mid \hat{\ell}} (1 + \frac{1}{p})} \Theta_{\hat{L}} \equiv_{\ell} \sum_{A \in \Gamma_0(\hat{\ell}) \backslash Sl(2,\mathbb{Z})} \Theta_{\hat{L}} \mid A \in M_k$$

hence need $s(\hat{L}) \in \bigcap_{p \equiv_{\ell} 0,-1} \mathbb{Z}_p^*$.

construction of $\hat{L}$:

example $r(L) = 2$:

$$\Theta_{[a,b,c]} = \sum q^{ax^2 + bxy + cy^2} = \sum q^{\frac{(2ax+by)^2}{4a} + \frac{\ell}{4a} y^2}$$

$$= \sum_{\substack{z,y \\ z \equiv by \bmod 2a}} q^{\frac{z^2}{4a} + \frac{\ell}{4a} y^2}$$

$$= \sum_{g \bmod 2a} \sum_{z \equiv_{2a} bg} q^{\frac{z^2}{4a}} \sum_{y \equiv_{2a} g} q^{\frac{\ell}{4a} y^2}$$

$$\equiv_{\ell} \sum_{g(2a)} \sum_{z \equiv_{2a} bg} q^{\frac{z^2}{4a}} \Big(\sum_{y \equiv_{2a} g} q^{\frac{y^2}{4a}} \Big)^{\ell} = \sum_{x \in \hat{L}} q^{\frac{x^2}{2}}$$

$\Theta_{\hat{L}^\sigma} = \Theta_{[a,b,c]}$

$\hat{L} = \left\{ \frac{1}{\sqrt{2a}} (z, y_1, \dots, y_\ell) \in \frac{1}{\sqrt{2a}} \mathbb{Z}^{\ell+1} \;\middle|\; y_1 \equiv_{2a} \dots \equiv_{2a} y_\ell,\ z \equiv_{2a} b y_1 \right\}$ $\quad \sigma: z \circlearrowleft,\ y_1 \to y_2 \to y_3 \to \dots \to y_\ell \to y_1$