① Oberwolfach

Often interested in situation:

Given $f \in M_k$ (mod forms on $SL(2,\mathbb{Z})$, wt k coeff. in $\mathbb{Z}$). Is f a theta series? ($f = \Theta_L = \sum_{x \in L} q^{x^2}$, with L)

Necessary criteria:

1/ $a_f(n) \geq 0$ for all n

This can in principle be decided on a computer:

Thm
f is a cusp form iff $a_f(n)$ changes sign ∞-ly many often.
If $a_f(0) > 0$ then is an explicitly calculable constant N_0 s.t. $a_f(n) \geq 0$ for all $n \geq N_0$.

So, in a sense, if 1/ is fullfilled this does not mean too much.

More meaningful: We can make a (short) list of primes ~~#~~ (possible prime divisors of #Aut(L)) and try to get information from this.
2/ f has to satisfy various special congruences:

Thm Suppose $f = \Theta_L = \sum_{x \in L} q^{x^2/2}$ for some (even, integral, unimod.) lattice L.
For $\ell \mid \#\mathrm{Aut}(L)$ a prime, let $\sigma \in \mathrm{Aut}(L)$ of order ℓ, then $\Theta_L \equiv \Theta_{L^\sigma} \mod \ell$.

Proof: For each n:
$$\{x \in L \mid \tfrac{x^2}{2} = n\} = \{x \in L^\sigma \mid \tfrac{x^2}{2} = n\} + \text{union of orbits w.r.t. } \langle\sigma\rangle \text{ with } \ell \text{ elements.} \quad \square$$

Suppose always $\ell \geq 5$

How to check for such congruences:

a/ $\underset{\text{level of } L^\sigma}{s(L^\sigma)} \mid \ell$ $\quad$ ($y \in (L^\sigma)^\#$, $x \in L$;
$$\ell y \cdot x = (1+\sigma+\dots+\sigma^{\ell-1})\, y \cdot x = y \cdot \underbrace{(1+\sigma+\dots+\sigma^{\ell-1})x}_{\in L^\sigma} \in \mathbb{Z}$$
hence $\ell y \in L^\# = L$)

b/ $\underset{\text{rank of } L}{r(L)} = r(L^\sigma) + n(\ell-1)$ with n odd if $d(L^\sigma) \neq \square$, n even if $d(L^\sigma) = \square$

(~~later~~) since $\mathrm{rank}(L^\sigma) = \deg \dfrac{\chi_\sigma(x)}{(x^{\ell-1}+\dots+1)^n}$, parity later)