An involution Assume $g\Gamma g = \Gamma$.

$g := \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ acts on $X_k(\Gamma)$ via

$$(g\lambda)(A) = g^{-1}\circ(\lambda(gAg))$$
$$= \lambda(gAg)(X,-Y).$$

[Note: $(g\lambda)(GA) = g\lambda(gGAg)$
$= g\; gGg\; \lambda(gAg) = G(g\lambda)(A)$]

$$X_k(\Gamma) = X_k(\Gamma)^+ \oplus X_k(\Gamma)^-$$

$k=2$: $X_k(\Gamma_0(N))^\pm = \{\lambda\in X_2(\Gamma_0(N)) : \lambda((x:y)) = \pm\lambda(-x:y)\}$

$X_2(\Gamma_0(p)) = \{\lambda : \lambda(x) = \pm\lambda(-x)\}$.

A simple not well-known fact concerning Hecke operators

Fix k, N

$\mathbb{T}$ = $\mathbb{C}$-Algebra spanned by all $T(\ell)\in \mathrm{End}(M_k(\Gamma_0(N)))$.

Thm Let X be a $\mathbb{T}$-module Hecke-equivariantly isomorphic to a subspace $M\subseteq M_k(\Gamma_0(N))$.
Then

$$M \doteq \Big\{\sum_{\ell\ge 1}(T(\ell)x, x^*)\, q^\ell \;\Big|\; x\in X,\ x^*\in X\Big\}$$

($\doteq$: $f,g\in\mathbb{C}[[q]]$, $f\doteq g$ iff $f = g + \text{const.}$).

Exercise: prove the Thm. Hint: for any $f\in M_k(\Gamma_0(N))$, any $n,\ell\ge 1$ one has $a_{T(\ell)f}(n) = a_{T(n)f}(\ell)$.