Hence by the Schur graph theory

$$\sum_i \left(\frac{f(x)}{p}\right) \overset{(a)}{=} \sum_{\substack{ad-bc=p\\ a>b\geq 0\\ d>c\geq 0}} \lambda\left([u:v]\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)$$

for a suitable $\lambda \in X_2(\Gamma)$, $[u:v] \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$

Manin reciprocity

Question • What is the meaning of λ in terms of E (or f)?
• Can one give an independent proof of (a) (Cartier?) using modular forms?

Example — table —

Problem 6 ~~Question~~: Can one construct an infinite family (λ_j) of Hecke eigensymbols $\lambda_j \in X_2(N_j)$ $(N_j \to \infty)$ such that λ_j does not map to Eisenstein series nor are "twists" of a given one?

Problem 7 The image of $X_2(N)$ in the cusp forms $\sum \lambda \mapsto \sum (T(\ell)\lambda)(\rho)\, q^\ell$ ~~space integer co~~ cusp forms $M_2(N)$ and consists of forms with integer coefficients. A Hecke eigenform (when suitably normalized) has algebraic coefficients. It follows that

- $K_f = \mathbb{Q}(a_f(\ell))\ (\ell = 1, 2, \ldots)$ is a # field of finite degree
- If $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, then $\sigma(f) := \sum \sigma(a_f(\ell))\, q^\ell \in M_2(N)$.

Question: What # fields occur as K_f? What other statements can one make about them?

— table —