Problem 4

Set $[r]_\ell := q^{-\frac{\ell}{2} B_2(\frac{r}{\ell})}$

$$\prod_{n \equiv r \,(\ell)} (1-q^n)^{-1} \prod_{n \equiv -r \,(\ell)} (1-q^n)^{-1}$$

$B_2(x) = y^2 - y + \frac{1}{6}$

$y = x - \lfloor x \rfloor$

This is a modular unit.

E := semi-group generated by the $[r]_\ell$.

special property:

span $\{[1]_5, [2]_5\}$ is $SL(2,\mathbb{Z})$-invariant.

span $\{\prod_{j \neq r} [j]_\ell \mid 1 \le r \le \frac{\ell-1}{2}\}$ is $SL(2,\mathbb{Z})$-inv.

Question: Characterise all finite subsets $S \subseteq E$ s.t. span (S) is $SL(2,\mathbb{Z})$-invariant. (partial results Eholzer – Skoruppa)

Problem 5

We learned in the opening lecture that modular forms represent reciprocity laws. I shall give here more evidence for this.

Let $E: y^2 = f(x)$ elliptic curve$/\mathbb{Q}$ (f cubic $\in \mathbb{Q}[x]$ without multiple zeroes)

For a prime p we have

$$a(p) := \#\{(x,y) \in \mathbb{F}_p \mid y^2 = f(x)\}$$
$$= \sum_{x \bmod p} 1 + \left(\frac{f(x)}{p}\right) = p + \sum_{x \in \mathbb{F}_p} \left(\frac{f(x)}{p}\right).$$

Wiles theorem: Let N be the conductor of E, $p \nmid N$. Then there is a Hecke eigenform $f \in S_2(N)$ such that

$$-\sum_{x \in \mathbb{F}_p} \left(\frac{f(x)}{p}\right) = p - a(p) = p\text{-th coefficient of } f.$$