Problem 3

Recall the Rogers–Ramanujan identities

$$q^{-\frac{1}{60}} \prod_{n \equiv \pm 1 \bmod 5} (1-q^n)^{-1} = \sum_{n \geq 0} \frac{q^{n^2 - \frac{1}{60}}}{(1-q) \cdots (1-q^n)}$$

$$q^{\frac{11}{60}} \qquad = \qquad q^{n^2+n+\frac{11}{60}}$$

These are generalized by the Andrews–Gordon identities:
For l odd $\in \mathbb{Z}_{\geq 1}$

$$\prod_{\substack{1 \leq i \leq \frac{l-1}{2} \\ i \neq r}} q^{-B_2(\frac{i}{l})/2} \prod_{\substack{n \equiv \pm i \bmod l \\ n > 0}} (1-q^n)^{-1}$$

$$= q^{-B_2(r/l)/2} \sum_{n \in \mathbb{Z}^{k-1}} \frac{q^{nAn^t + b_r n^t}}{(q)_{n_1} \cdots (q)_{n_{k-1}}}$$

$(q)_m = (1-q) \cdots (1-q^m)$. $\left(k = \frac{l-1}{2},\ n = (n_1, \ldots, n_{k-1})\right)$

$A = (\min(i,j)) \qquad b_r = (\min(i+1-r, 0) \ldots)$

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The special property here is that the left hand sides are so-called modular units – special modular functions on certain congruence subgroups. Hence

Question: For what symmetric $A \in \mathbb{Q}^{n \times n}$, $b \in \mathbb{Q}^n$, $c \in \mathbb{Q}$ is

$$f_{A,b,c} := \sum_{r_1, \ldots, r_n \in \mathbb{Z}} \frac{q^{rAr^t + br^t + c}}{(q)_{r_1} \cdots (q)_{r_n}}$$

a modular function?

Answer only known for $n = 1$. $f_{A,b,c}$ is modular for

$A\ b\ c$ | — table — . (Zagier)