Concluding Remarks

Problem 1 09-08-2010

We have M_k is generated by $E_4^a E_6^b$, $4a+6b=k$, hence

$$\sum \dim M_k X^k = \sum_{a,b} X^{4a+6b} = \frac{1}{(1-X^4)(1-X^6)}$$

If $\Gamma \underset{f.i.}{\subseteq} SL(2,\mathbb{Z})$, one defines

$$P_\Gamma := \sum \dim M_k(\Gamma)\, X^k \qquad \text{(Hilbert Poincaré series).}$$

It is not hard to prove (see my Allahabad lectures from last year):

$$P_\Gamma = \frac{p_\Gamma(X)}{(1-X^4)(1-X^6)}, \quad \text{where } \underset{=1+\dots}{p_\Gamma(X)} \in \mathbb{Z}[X]_{\deg \le 12}.$$

— table —

The reason for this is (see Allahabad lectures)

Thm $M_*(\Gamma) = \bigoplus_{k\ge 0} M_k(\Gamma)$ is a free $M_* := \bigoplus M_k$ module of rank $[SL(2,\mathbb{Z}):\Gamma]$ (say if $-1\in\Gamma$).

In fact, if the $f_1, \dots, f_n$ form a M_*-basis, say $wt(f_j) = k_j$, then

$$M_k(\Gamma) = M_{k-k_1} f_1 \oplus \dots \oplus M_{k-k_n} f_n$$

and

$$P_\Gamma = \sum \dim M_{k-k_1} X^k + \dots + \sum \dim M_{k-k_n} X^k = \frac{X^{k_1} + X^{k_2} + \dots + X^{k_n}}{(1-X^4)(1-X^6)}.$$

In particular, $p_\Gamma = 1 + a_1 X + \dots + a_{12} X^{12}$, $a_k = \#$ of f_i of wt k.

The theorem also shows

Corollary $M_*(\Gamma)$ is finitely generated as algebra over $\mathbb{C}$,