Now (looking up in Niwa's paper) one can work out

$$\vartheta = \sum_{\mathcal{D}\in\mathbb{Z}} q^{\mathcal{D}} \sqrt{\operatorname{Im}(\tau)} \sum_{\substack{Q=[a,X,?] \\ \in F_N(4Nt\mathcal{D})}} \underbrace{\chi\left(\tfrac{a}{N}\right)\left(\tfrac{-4t}{a/N}\right)}_{\chi_t(Q)} \frac{Q(\bar z)}{\operatorname{Im}(z)} e^{-\frac{\pi}{Nt}\operatorname{Im}(\tau)\frac{\hat Q(z)^2}{\operatorname{Im}(z)^2}} dz$$

$$F_N(\Delta) = \left\{ aX^2+bX+c \in \mathbb{Z}[X] \;\middle|\; \begin{matrix} N|a \\ 2N|b \end{matrix} \quad b^2-4ac=\Delta \right\}$$

$$\hat Q(z) = a|z|^2 + b\operatorname{Re}(z) + c$$

$$(q = e^{2\pi i\tau})$$

By some simple transformations

$$\int_0^\infty \vartheta\circ A + \vartheta\circ A^*$$

$$= \sum_{\mathcal{D}\in\mathbb{Z}} q^{\mathcal{D}} \sqrt{\operatorname{Im}\tau} \int_0^\infty \sum_{\substack{Q\in F_N(4Nt\mathcal{D})\circ A \\ \shortparallel \\ aX^2+bX+c}} \chi_t(Q\circ A^{-1}) \left(at-\frac{c}{t}\right) e^{-\frac{\pi}{Nt}\operatorname{Im}(\tau)\left(at+\frac{c}{t}\right)^2}$$

If $4Nt\mathcal{D} \neq \square$, then $ac\neq 0$ and

$$\int_0^\infty \left(at-\frac{c}{t}\right) e^{-\frac{\pi}{Nt}\operatorname{Im}(\tau)\left(at+\frac{c}{t}\right)^2} = \frac{1}{2}\left(\operatorname{sign}(a) - \operatorname{sign}(c)\right) \times \sqrt{\frac{Nt}{\operatorname{Im}\tau}}$$

<u>Note</u> $\ne 0 \iff ac<0 \implies \mathcal{D}>0$

~~Setting~~ 2) $S\mathcal{D}$, for $(p:q)\in\mathbb{P}^1(\mathbb{Q})$, $(p,q)=1$, $p,q\in\mathbb{Z}$:

$$Q(p:q) = ap^2+bpq+cq^2$$

Then $a = Q(\infty)$, $(=Q(1:0))$, $c = Q(0)$ $(=Q(0:1))$, $aX^2+bX+c = Q\circ A$ then $\operatorname{sign}(a)-\operatorname{sign}(c) = \operatorname{sign}(Q(A\infty)) - \operatorname{sign}(Q(A0))$

<u>Theorem</u> The $\mathcal{D}$-th Fourier coefficient of $L_{t,\chi}\left(\sum_s c_s(s)\right)$, ~~for $4Nt\mathcal{D}\neq\square$~~, nonsquare part of $\mathcal{D} \neq 4Nt$, is given by

$$\sum_{Q\in F_N(4N\mathcal{D}t)} \chi_t(Q) \sum_s c_s \operatorname{sign}(Q(s))$$