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One can go through all these maps and finds injections:

Cor. $\mathcal{M}_k(\Gamma)^{\pm}_{\mathbb{C}} \longleftrightarrow S_k(\Gamma)$ ~~$\oplus \mathcal{E}_k(\Gamma)$~~ $: \Lambda^{\pm}$

The application from Λ_f: $\Lambda(f) =: \Lambda_f$, $\Lambda_f \left(\sum c_\nu e_\nu\right) = \sum c_\nu \int_\gamma^\infty f(z)(X - zY)^{k-2}\,dz$

defines injection, when $\Lambda^{\pm}$ are Λ followed by the projection ~~then~~ $\mathcal{M}_k(\Gamma)^{\pm}_{\mathbb{C}}$ only $\mathcal{M}_k(\Gamma)^{\pm}_{\mathbb{C}}$.

Here $\mathcal{M}_k(\Gamma)^{\pm}_{\mathbb{C}}$ eigenspaces of the involution

$$(\Sigma\lambda)(C) := g\,\lambda(gC).$$

$\mathbb{T}$ acts on $\mathcal{M}_k(N) := \mathcal{M}_k(\Gamma_1(N))$ as follows

$$(T(\ell)\lambda)(C) := \sum_{\substack{M \in \Gamma_1(N)\backslash \Delta_1(N) \\ \det M = \ell}} M^{-1}\cdot\lambda(MC).$$

Thm For $\varepsilon \in \{\pm 1\}$ $\mathcal{M}_k^{\varepsilon}(N)$ is isomorphic as Hecke module to a ~~subspace~~ ~~of $M_k(N)$, which contains all~~ $S_k(N) \oplus \mathcal{E}_k^{\varepsilon}(N)$, where $\mathcal{E}_k^{\varepsilon}(N)$ is a subspace in $\mathcal{E}_k(N)$ s.th. $\mathcal{E}_k(N) = \mathcal{E}_k(N)^{+} \oplus \mathcal{E}_k^{-}(N)$.

Verify by hand ~~Pf~~ $\Lambda^{\varepsilon}: S_k(N) \to \mathcal{M}_k^{\varepsilon}(N)$ ~~also~~ ~~is~~ $\mathbb{T}$-linear.

Describe explicitly the Hecke eigenforms in $\mathcal{M}_k^{\varepsilon}(N)$ which correspond to Eisenstein series. $\square$

Consequences

Thm ~~For each $\varepsilon \in \pm$~~ For each ~~$\lambda \in \mathcal{M}_k(N)$,~~ ~~$C \in \mathbb{Z}[\Gamma\backslash \ldots]$~~ the application

$$S_C : \lambda \longmapsto \sum_{\ell=1}^{\infty} (T(\ell)\lambda)(C)\,q^{\ell}$$

~~defines~~ defines a $\mathbb{T}$-module morphism $\mathcal{M}_k(N) \longrightarrow M_k(N)$.

One has $\bigcup_C \operatorname{Im} S_C = M_k(N)$.

Ex ~~Show that~~ Find good $T(\ell)$ on $X_k(N)$ s.t. $\mu^*: \mathcal{M}_k(N) \to X_k(N)$ is $\mathbb{T}(N)$-~~linear~~. For $k=2$ (see above).