Pf of Thm

$\operatorname{Im}(\mu^*) \subseteq \mathcal{M}_k(\Gamma)$ since

$$\lambda(e_{A\infty} - e_{A0}) + \lambda(e_{AS\infty} - e_{AS0}) = 0$$

$$\lambda(e_{\underset{\infty}{A\infty}} - e_{\underset{0}{A0}}) + \lambda(e_{\underset{0}{AR\infty}} - e_{\underset{-1}{AR0}}) + \lambda(e_{\underset{-1}{AR^2\infty}} - e_{\underset{\infty}{AR^20}}) = 0$$

Manin's Lemma $\Rightarrow$ μ^* injective.

Surjectivity follows from

$$\dim_{\mathbb{C}} X_k(\Gamma)_{\mathbb{C}} = \dim_{\mathbb{C}} \mathcal{M}_k(\Gamma)_{\mathbb{C}}$$

see the proved thm. above and the new thm. $\square$

A bit homological algebra:

short exact sequence: $\quad (V_k = \mathbb{C}[X,Y]_{k-2})$

$$0 \to V_k \xrightarrow{\iota} \operatorname{Hom}(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})], V_k) \xrightarrow{\text{res.}} \operatorname{Hom}(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0, V_k) \to 0$$

$$f \mapsto \text{const map } f$$

of Γ-modules.

Long exact sequence:

$$V_k^\Gamma \to \mathcal{E}_k(\Gamma)_{\mathbb{C}} \to \mathcal{M}_k(\Gamma)_{\mathbb{C}} \xrightarrow{\delta} H^1_*(\Gamma, V_k) \xrightarrow{\iota_*} H^1_*(\Gamma, \operatorname{Hom}(\mathbb{Z}[\ldots], V_k))$$

$$\cup\!| \qquad H^1_{\text{cusp}}(\Gamma, V_k) := \operatorname{Ker} \iota_*$$

Eichler–Shimura cohomology (cuspidal cohomology)

Hence

Lemma $\quad \mathcal{M}_k(\Gamma)_{\mathbb{C}} / \operatorname{im}(\mathcal{E}_k(\Gamma)_{\mathbb{C}}) \cong H^1_{\text{cusp}}(\Gamma, V_k)$

Thm. (Eichler–Shimura) The map $f \oplus g \mapsto$ "$G \mapsto \int_z^{Gz} f(z)(X - zY)^{k-2} + g(z)(X - \bar{z}Y)\,d\bar{z}$" induces an isom.

$$H^1_{\text{cusp}}(\Gamma, V_k) \cong S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$$

Ex Prove $\dim \mathcal{E}_k(\Gamma) = \#_\Gamma\backslash\mathbb{P}^1(\mathbb{Q}) - \delta(k=2)$.

Consequence: 1) $\dim \mathcal{M}_k(\Gamma) = \dim \mathcal{E}_k(\Gamma) + 2 \dim S_k(\Gamma)$.