§3 Eichler-Shimura-Cohomology and modular symbols

Set, for $\Gamma \underset{f.i.}{\leq} SL(2,\mathbb{Z})$ (not nec. $-1 \in \Gamma$):

$$\mathcal{M}_k(\Gamma) := \mathrm{Hom}_\Gamma(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{Z}[X,Y]_{k-2})$$

restriction map

$$\mathcal{E}_k(\Gamma) := \mathrm{Hom}_\Gamma(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})], \mathbb{Z}[X,Y]_{k-2}).$$

Action on $\mathbb{Z}[X,Y]$, $(A,f) \mapsto f(A^{-1}\binom{X}{Y})$.

$\mathcal{M}_k(\Gamma)_\mathbb{C} := \mathbb{C} \otimes_\mathbb{Z} \mathcal{M}_k(\Gamma)$ = same def. with $\mathbb{Z}[X,Y]$ repl. by $\mathbb{C}[X,Y]$.

$$X_k(\Gamma) := \left\{ \lambda : SL(2,\mathbb{Z}) \to \mathbb{C}[X,Y]_{k-2} \;\middle|\; \begin{array}{l} \lambda(GX) = G\cdot\lambda(X) \\ \lambda(X) + \lambda(XS) = 0 \\ \lambda(X) + \lambda(XR) + \lambda(XR^2) = 0 \end{array} \right\}$$

Note: $X_k(\Gamma)$ is redefined here, but can be identified with the ~~labelling~~ $\mathbb{Z}$-module of admissible labels of G_Γ as before.

Obvious map $\mu: SL(2,\mathbb{Z}) \to \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0$, $A \mapsto e_{A\infty} - e_{A0}$

$\mu^*: \mathcal{M}_k(\Gamma) \longrightarrow X_k(\Gamma)$, $\lambda \longmapsto \mu^*\lambda$,

$(\mu^*\lambda)(A) = \lambda(e_{A\infty} - e_{A0})$.

Thm The map μ^* is an isomorphism of $\mathbb{Z}$-modules.

Lemma (Manin)

$\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0$ is a free $\mathbb{Z}[SL(2,\mathbb{Z})]$-module of rank 1.

Pf. $\sum c(r)\, e_r = \sum c(r)(e_r - e_\infty)$

$$e_r - e_\infty = e_{\frac{p_1}{q_1}} - e_\infty + e_{\frac{p_2}{q_2}} - e_{\frac{p_1}{q_1}} + \dots + e_{\frac{p_n}{q_n}} - e_{\frac{p_{n-1}}{q_{n-1}}}$$

where $\frac{p_j}{q_j}$ are the convergents of cont. frac. exp. of $r \left(= \frac{p_n}{q_n}\right)$ $\left(\frac{p_0}{q_0} = \frac{1}{0}\right)$. Fact: $A_j := \begin{pmatrix} p_n & p_{n-1}(-1)^j \\ q_n & q_{n-1}(-1)^j \end{pmatrix} \in SL(2,\mathbb{Z})$ $\square$

Hence $e_r - e_\infty = (A_1 + A_2 + \dots + A_n)(e_\infty - e_0)$.

[if $r = [a_0, \dots, a_n]$, then $A_j = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-1)^{j-1} \end{pmatrix}$]