<u>Remark</u> The action of $T(\ell)$ on $X_2(N)$ can be described in a more pleasant form (Merel):

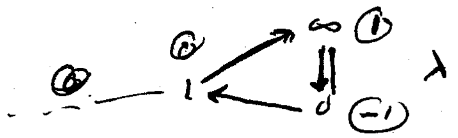
$\mathcal{M}_\ell := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} ;\ a > b \geq 0 \quad d > c \geq 0 \right\}$ ~~ad-bc=ℓ~~

$\mathcal{M}_\ell := \{ A \in \mathcal{M} \mid \det A = \ell \}$

Thm
$$(T(\ell)\lambda)(v) = \sum_{M \in \mathcal{M}_\ell} \lambda(vM).$$

$\boxed{\begin{array}{l} \text{Thm} \\ M_2(N) \text{ is spanned by the series} \\ \displaystyle\sum_{\substack{a \geq b > 0 \\ d \geq c > 0}} \lambda\left(v\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) q^{ad-bc} \\ \text{when } \lambda \in X_2(N),\ v \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \end{array}}$

<u>Ex.</u> $\Gamma_0(p)$



$$\sum_{M \in \mathcal{M}_\ell} \lambda((0:1)M) = \sum_{M \in \mathcal{M}_\ell} \lambda(c:d) = \sum_{\substack{a > b \geq 0 \\ d > 0 \quad ad = \ell \\ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \\ (d,N)=1}} (-1) + \cancel{\sum_{\substack{a > b \\ d \geq c > 0}}} = -\sum_{\substack{d \mid \ell \\ (d,N)=1}} \frac{\ell}{d}$$

$$E := -\sum T(\ell)\lambda(0:1)\, q^\ell = +\sum_{\ell \geq 1}^{\infty} \sum_{\substack{a \mid \ell \\ (\frac{\ell}{a}, N)=1}} a\, q^\ell$$

<u>Ex</u> Show that $E(z) = \sum_{t \mid N} \mu(t)\, t\, E_2(tz)$

<u>Ex</u> $\Gamma_0(11)$: ... $\quad X_2(11)^-_0 = \mathbb{C}\cdot\lambda \quad \lambda(a) = \begin{cases} 1 & a \equiv_{11} 3,4 \text{ say} \\ -1 & a \equiv_{11} -3,-4 \\ 0 & \text{otherwise} \end{cases}$

Hence

$$\eta(z)^2 \eta(11z)^2 = \text{const.} \sum_{\substack{a \geq b \geq 0 \\ d > c \geq 1 \\ (c,11)=1}} \left[\delta\left(3\tfrac{a}{c} \equiv_{11} 3,4\right) - \delta\left(3\tfrac{a}{c} \equiv_{11} -3,-4\right) \right] q^{ad-bc} \qquad \left((3:1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (3a : c) \right)$$

$$\sum \left(\frac{12}{rstu}\right) q^{\frac{r^2+s^2+11t^2+11u^2}{24}}$$

$$= \text{const}\left(\sum_{\substack{a > b \geq 0 \\ d > c \geq 1 \\ \frac{3a+c}{3b+d} \equiv_{11} 3,4}} q^{ad-bc} - \sum_{\equiv_{11} -3,-4} q^{ad-bc} \right)$$

Ex*: Find an elementary pf for the identity.

go to p. 10