



An involution Assume $g \Gamma g = \Gamma$.

$g := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts on $X_n(\Gamma)$ via

$$\begin{aligned} (g\lambda)(xA) &= g^{-1} \cdot (\lambda(gAg)) \\ &= \lambda(gAg)(x, -y). \end{aligned}$$

$$\begin{aligned} [\text{Note: } (g\lambda)(gA) &= g\lambda(gAg) \\ &= g \cdot g\lambda(gAg) = g(g\lambda)(A)] \end{aligned}$$

$$X_n(\Gamma) = X_n(\Gamma)^+ \oplus X_n(\Gamma)^-$$

$$h=2: X_n(\Gamma_0(N))^{\pm} = \{ \lambda \in X_n(\Gamma_0(N)) : \lambda(gx) = \pm \lambda(-x) \}$$

$$X_n(\Gamma_0(N)) = \{ \lambda : \lambda(x) = \pm \lambda(-x) \}.$$

simple not well-known fact concerning Hecke operators

Fix n, N

$\mathbb{T} = \mathbb{C}$ -Algebra spanned by all $T(\ell) \in \text{End}(M_n(\Gamma_0(N)))$.

Thm Let X be a \mathbb{T} -module Hecke-equivariantly isomorphic to a subspace $M \subseteq M_n(\Gamma_0(N))$.

Then

$$M \subseteq \left\{ \sum_{\ell \geq 1} (T(\ell)x, x^*) q^\ell \mid x \in X, x^* \in X \right\}$$

$$(\Leftrightarrow: f, g \in \mathbb{C}[[q]], f \equiv g \text{ iff } f = g + \text{const.})$$

Exercise: prove the Thm. Hint: for any $f \in M_n(\Gamma_0(N))$, any $m, \ell \geq 0$ one has $a_{T(m)} f(n) = a_{T(m)} f(\ell)$.