admissible labelings of degree n: $\lambda : SL(2,\mathbb{Z}) \longrightarrow \mathbb{Z}[X,Y]_n$

$\lambda(GX) = G\cdot\lambda(X)$

s.t. $\lambda(X) + \lambda(XS) = 0$

$\lambda(X) + \lambda(XR) + \lambda(XR^2) = 0$

$X_k(\Gamma) = \mathbb{Z}$-module of admissible labelings of degree $k-2$. (Relation to labelling before: $\lambda \in X_k(\Gamma)$ corresponds to $v \mapsto (A, \lambda(A))$, (for $A \in v$))

Note $X_2(\Gamma)$ newly defined can be identified with old $X_2(\Gamma)$ in obvious manner.

Example $\mathcal{G}_{SL(2,\mathbb{Z})}$:

$X_2(SL(2,\mathbb{Z}))$: $\lambda : {}_{SL(2,\mathbb{Z})}\backslash SL(2,\mathbb{Z}) \to \mathbb{Z}$ (i.e. λ const.)

$(\lambda + \lambda)\ \lambda = 2\lambda = 0$, i.e. $\lambda = 0$

$X_k(SL(2,\mathbb{Z}))$: $\lambda : SL(2,\mathbb{Z}) \to \mathbb{Z}[X,Y]_{k-2}$ $\quad (X,Y)\begin{pmatrix}0&1\\-1&0\end{pmatrix} = (-Y,X)$

$\lambda(X) = X\cdot\lambda(1)$

$\lambda(1) + S\lambda(1) = 0$ $\quad (X,Y)R^{-1} = (X,Y)\begin{pmatrix}1&1\\-1&0\end{pmatrix}$

$\lambda(1) + R\lambda(1) + R^2\lambda(1) = 0$ $\quad R^2 = \begin{pmatrix}0&-1\\1&1\end{pmatrix}$, $= (X-Y, X)$

i.e. $\lambda(1) \Rightarrow f : f \in \mathbb{Z}[X,Y]_{k-2}$: $f(X,Y) + f(-Y,X) = 0$

$\cancel{f(X,Y) + f(Y, Y-X) + f(-X+Y, -X)} = 0$

$\Leftarrow X_k(SL(2,\mathbb{Z}))_0$ $f(X,Y) + f(X-Y, X)$ $+ f(-Y, X-Y) = 0$

Exercises: 1) Determine the first $X_k(SL(2,\mathbb{Z}))$

2) Prove $\dim X_k(SL(2,\mathbb{Z})) = 2\dim S_k(SL(2,\mathbb{Z}))$. $+\,\mathrm{dr}$

~~Submit before next Thursday.~~

Remark

$f(X,Y) := X^n - Y^n$, n even is always in $X_n(SL(2,\mathbb{Z}))$.

becomes $(X^2 - Y^2) + ((X-Y)^4 - X^4) + ((-Y)^4 - (X-Y)^4) = 0$