Thm Suppose X is a $\mathbb{T}(N)$-module isomorphic to a $\mathbb{T}$-invariant subspace S of $M_k(N)$. Then

$$\rho(S) = \Big\{ \sum_{\ell \geq 1} \lambda(T(\ell)x)\, q^\ell \;\Big|\; \begin{matrix}\lambda \in X^* \\ x \in X\end{matrix} \Big\}.$$

Here $\rho : M_k(N) \to \mathbb{C}[[q]]$, $f \mapsto \sum_{\ell \geq 1} a_f(\ell)\, q^\ell = f - a_f(0)$.

Rmk X + $\mathbb{T}(N)$-action explicit $\Rightarrow$ $M_k(N)$ explicit

Pf. $\lambda_n \in S^*$, $\lambda_n(f) = a_f(n)$ span S^*
(since $\lambda_n(f) = 0$ for all n implies $f \equiv 0$).
Let $\tilde{\lambda}_n = \varphi^* \lambda_n$ where $\varphi : X \xrightarrow{\sim} S$. Thus the $\tilde{\lambda}_n$ span X^*. Set $F_{\lambda,x} = \sum_{\ell \geq 1} \lambda(T(\ell)x)\, q^\ell$ $(\lambda \in X^*)$.

It suffices to show
1) $F_{\tilde{\lambda}_n, x} \in \rho(S)$,
2) $F_{\tilde{\lambda}_1, \varphi^{-1}(f)} = f$

For 1):
$$F_{\tilde{\lambda}_n, x} = \sum \lambda_n(T(\ell)\varphi(x))\, q^\ell$$
$$= \sum \lambda_n(T(\ell)f)\, q^\ell \qquad f := \varphi(x)$$
$$= \sum a_{T(\ell)f}(n)\, q^\ell$$
$$= \sum a_{T(n)f}(\ell)\, q^\ell = T(n)f.$$

For 2) Clear from 1). $\square$

~~**Ex.** $(T, z) \mapsto (\deg_k T)\, z$ defines an action of $\mathbb{T}(1)$ on $\mathbb{C}$.~~
~~$\deg_k T = \sum c(x) \det(x)^{k-1}$ $\big(T = \sum_{x \in \Gamma_1 \backslash C_1} c(x)\, e_x\big)$~~

~~$\mathbb{C} \cdot E_k(1) \to X$, $f \mapsto a_f(0)$ is $\mathbb{T}$-module isomorphism.~~
~~$E_k - a_{E_k}(0) = \sum (\deg_k T(\ell))\, q^\ell = \sum \big(\sum_{d \mid \ell} d^{k-1}\big)\, q^\ell.$~~