Exercise: 2) A set of representatives for $\Gamma_0(N)\backslash G_0(N)$ is given by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad \underset{\substack{a\geq 1 \\ \gcd(a,N)=1}}{ad=N}, \quad b \bmod d$$

$$\left(\cancel{\text{Fact: } T(\ell)T(m) = \sum_{d\mid \ell,m} T\left(\frac{\ell m}{d^2}\right) ?} \right)$$

~~H = H(N) = subalgebra of H generated spanned by the T(ℓ), ℓ ∈ Z≥1.~~

$GL(2,\mathbb{R}) \times Hol(\mathfrak{H}) \to Hol(\mathfrak{H}) : (A, f) \mapsto f|_k A := (c\tau+d)^{-k} \det(A)^{k/2} f(Az)$

Thm 1) H acts on $M_k(N)$ via $(\det X)^{k-1}$

$$(T, f) \longmapsto Tf := \sum_{X \in \Gamma_0(N)\backslash G_0(N)} c(X)\, f|_k X \qquad \left(T = \sum c(X)\, e_X\right)$$

2) One has

$$T(\ell) f := \sum_{n \geq 0} \left(\sum_{d \mid \ell, n} d^{k-1} a_f\left(\frac{\ell n}{d^2}\right) \right) q^n.$$

Remark $\boxed{a_{T(\ell)f}(n) = a_{T(n)f}(\ell)}$

pf. (of (2))

$$\ell^{1-k}\, T(\ell) f = \sum_{\substack{ad=N \\ b \bmod d \\ \gcd(a,N)=1}} f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum \frac{(ad)^{k/2}}{d^k} f\left(\frac{a\tau+b}{d}\right)$$

d^{-k}

$$= \sum \left(\frac{a}{d}\right)^{k/2} \sum_n a_f(n)\, q^{an/d}\, e\left(\frac{bn}{d}\right)$$

$$= \sum \left(\frac{a}{d}\right)^{k/2} d^{1-k} \sum_{\substack{n \\ d \mid n}} a_f(n)\, q^{an/d}$$

ℓ^{1-k}

$$= \ell^{\cancel{k/2}} \sum_{\substack{a \mid n \\ \gcd(a,N)=1}} a^{k-1} \sum_m a_f(dm)\, q^{am}$$

$$= \cancel{\ell^{1-k/2}}\ \ell^{1-k} \sum_n \left(\sum_{\substack{d \mid \ell, n \\ \gcd(a,N)=1}} a^{k-1} a_f\left(\frac{\ell m}{a^2}\right) \right) q^n \qquad \square$$