Chapter I: MF of integral wt

Notations

$SL(2,\mathbb{Z}) = \ldots$

$\Gamma_0(N) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap SL(2,\mathbb{Z})$

$M_k(N)$ = space of MF on $\Gamma_0(N)$ $(k \in \mathbb{Z})$

$= \{ f : \mathfrak{H} \xrightarrow{hol} \mathbb{C} \mid 1)\ f(A\tau) = (c\tau+d)^k f(\tau)\ \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$
$\quad 2)\ f$ hol. at the cusps $\}$

$S_k(N)$ = subspace of cusp forms in $M_k(N)$

$E_k(N)$ = subspace of Eisenstein series

<u>Fact</u>: $M_k(N) = S_k(N) \oplus E_k(N)$

$T(\ell)$ = ℓ-th Hecke operator on $M_k(N)$ (see below). $(\ell \in \mathbb{Z}_{\geq 1})$

<u>Fact</u>: Every $f \in M_k(N)$ possesses a FE of the form

$$f = \sum_{n \geq 0} a_f(n)\, q^n.$$

If $f \in S_k(N)$, then $a_f(0) = 0$.

<u>Exercise</u>: 1) Show that $\Gamma_0(N)$ is a subgroup

$\mathbb{C}[X] = \{ f : X \to \mathbb{C} \}$ is a $\mathbb{C}$-vector space

basis e_x $(x \in X)$, where $e_x(y) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise} \end{cases}$.

§2 Hecke-modules and modular forms of integral weight

$G_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \;\middle|\; \begin{array}{l} ad - bc > 0 \\ N \mid c,\ \gcd(a,N) = 1 \end{array} \right\}$ semi-subgroup of $GL(2,\mathbb{Q})$

Note $G_0(N)$ inv. under right and left multiplication with $\Gamma_0(N)$.

$\mathcal{H} = \mathcal{H}(N) = \mathcal{H} := H(G_0(N), \Gamma_0(N)) = \{ \mathcal{H} \in \mathbb{C}[\Gamma_0(N) \backslash G_0(N)] : \mathcal{H} = \mathcal{H}A\ \forall A \in \Gamma_0(N) \}$

$T(\ell)\ \mathcal{H} := \sum_{A \in \Gamma_0(N) \backslash G_0(N)_{\det = \ell}} e_A \in \mathcal{H}$ $\quad \left(\left(\sum_{c \in \Gamma_0(N)A}\right) \left(\sum_{d \in D^{-1}\Gamma_0(N)D}\right) = \sum_{c \in CA^{-1}d(D^{-1}\Gamma_0(N)AB)} \right)$

<u>Fact</u> $\mathcal{M}(N) = \cancel{\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \mid N \mid c,\ \gcd(a,N) = 1 \right\}}$.

$T(\ell)$ generate $\mathcal{H}(N)$ as $G_0(N)$ abelian