Weil representation associated to $\underline{L}$

Assume r even.

Fact: $\Theta_{\underline{L}} := \mathrm{span}_{\mathbb{C}} \{ \theta_{\underline{L}}|_{r/2} A \mid A \in SL(2,\mathbb{Z})\}$ $\quad (f|_k A = f(Az)(cz+d)^{-k})$

$= \mathrm{span}_{\mathbb{C}}\{\theta_{x+\underline{L}} : x+L \in L^\#/L\}$ $\quad (\theta_{x+\underline{L}} = \sum_{r\in x+L} q^{Q(r)})$

Hence

$\underline{L} \rightsquigarrow SL(2,\mathbb{Z})$-right module $\Theta_{\underline{L}}$

Not quite the Weil representation. Define

$$\vec{\vartheta}_{\underline{L}} : \mathfrak{H}\times \mathbb{C}\otimes L \longrightarrow \mathbb{C}[L^\#/L] \quad \left(= \text{formal C-lin. comb. of symbols } \mathfrak{e}_\xi,\ \xi\in L^\#/L\right)$$

$$\vec\vartheta_{\underline{L}}(z,w) = \sum_{x\in L^\#} q^{Q(x)} e^{2\pi i B(x,w)} \mathfrak{e}_{x+L}$$

Then for all $A\in SL(2,\mathbb{Z})$ there is a $\omega(A)\in GL(\mathbb{C}[L^\#/L])$ such that

$$\vec\vartheta_L|A \quad \left(= \vec\vartheta_{\underline{L}}\left(Az, \frac{w}{cz+d}\right) e^{-2\pi i c \frac{Q(w)}{cz+d}}\right)$$

$$= \omega(A)\,\vec\vartheta_L \qquad \text{(e.g. Kloostermann)}$$

Def. $\omega_{\underline{L}} = \omega$ Weil representation ass. to $\underline{L}$

Easy to see: $\omega_{\underline{L}}$ depends only on $\mathfrak{D}_{\underline{L}}$.

Invariants: $\mathfrak{D}_{\underline{L}}$, $\omega_{\underline{L}}$, $\Theta_{\underline{L}}$

What are the relations among these?

More generally:

$\underline{A} = (A, Q)$ f.q.m. $\rightsquigarrow$ $\omega_{\underline{A}}$ Weil rep. of $SL(2,\mathbb{Z})$ on $\mathbb{C}[A]$

$SL(2,\mathbb{Z}) = \langle S,T\rangle$

$\omega_{\underline{L}} = \omega_{\mathfrak{D}_{\underline{L}}}$

$T e_x = e_x\, e(Q(x))$

$S e_x = \sigma(A)\frac{1}{\sqrt{|A|}}\sum_{y\in A} e(-B(x,y))\, e_y$