同濟大學 教师备课用纸

Useful since:

* these are automorphic functions and hence theory of automorphic function may be applied to obtain results about lattices. / Good classification of lattices, existence of lattices with prescribed densities etc /

Theta series clearly an invariant of $\underline{L}$, i.e.: [appl. of rep. #'s]

$\underline{L} \cong \underline{M}$ then $\Theta_{\underline{L}}^{(n)} = \Theta_{\underline{M}}^{(n)}$, $\vartheta_{\underline{L}} = \vartheta_{\underline{M}}$ ...

(i.e. if ex. $\phi: L \to M$ isom of gps s.t. $B_1(x,y) = B_2(f(x), f(y))$ ($\underline{L} = (L, B_1)$, $\underline{M} = (M, B_2)$))

Easy to see:

$\Theta_{\underline{L}}^{(r)} = \Theta_{\underline{M}}^{(r)}$ ($r = \mathrm{rank}(\underline{L}) = \mathrm{rank}(\underline{M})$)

then $\underline{L} \cong \underline{M}$ ($L = \mathbb{Z}e_1 + \dots + \mathbb{Z}e_n$, $S = (\tfrac{1}{2} B(e_i, e_j))$) $\quad A_{\underline{L}}(T) = \#\{X \in \mathbb{Z}^{r\times r} \mid X^t S X = 2T\}$

~~[in fact: S, T Gram matrices of $\underline{L}, \underline{M}$, then~~ and

~~$\Theta_{\underline{L}}^{(r)} = \Theta_{\underline{M}}^{(r)}$ implies $A_{\underline{L}}(T) = A_{\underline{M}}(T) > 0$,~~

~~i.e. $M^t S M = T$ for a $M \in \mathbb{Z}^{r\times r}$~~

~~and similarly $N^t T N = S$ for $N \in \mathbb{Z}^{r\times r}$.~~

~~It follows $\det M = \det N = \pm 1$ ... $\xrightarrow{\phi}$ ...]~~

~~However $\Theta_{\underline{L}}^{(n)}$ quite useless as invariant since we are not really able to do explicit calculations with Siegel modular forms of high degree. But $\Theta_{\underline{L}}$ useful since although not~~

~~How "much" of $\underline{L}$ is determined by $\Theta_{\underline{L}}$?~~ very well with these.

Ex 1) Famous examples:

$$\Theta_{E_8 \oplus E_8} = \Theta_{E_{16}}$$

($E_8 \oplus E_8$, E_{16} the degree of unimodular even lattices / iso.; simply since corresp. space of modular forms is 1-dim.)

Ex $\underline{L} = (L, B)$ unimodular ($\det B(e_i, e_j) = 1$)

Consider the $r \equiv 0 \bmod 8$

$\Theta_{\underline{L}} \in M_{r/2} = \mathrm{span}_{\mathbb{C}} \langle E_4^a E_6^b \mid 4a + 6b = r/2,\ a, b \geq 0 \rangle$, $E_4 = 1 + 240 \sum_{n\geq1} \sigma_3(n) q^n$

$M_4 = E_4 \quad M_8 = E_4^2 \quad M_{12} = E_4^3, E_6^2$ ... $\quad E_6 = 1 - 504 \sum_{n\geq1} \sigma_5(n) q^n$

# unimod. lattices	8	16	24	32
	1	2	24	$> 80 \cdot 10^6$
	E_8	E_{16}, $E_8 \oplus E_8$		